DYNAMIC ROAD PRICING WITH TRAFFIC-FLOW DEPENDENT TOLLING

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ABSTRACT

In this paper the dynamic optimal toll design problem as a game of the Stackelberg type is investigated, with the road authority as a leader and drivers of the road network as followers. The road authority sets tolls on some links in the network such as to minimize total travel time of the system, while in each time instance the travelers choose their routes so as to minimize their own perceived travel costs. We assume that only proper subset of links can be tolled (second-best tolling). Two types of problems are studied: The “classical” Stackelberg game with the road authority imposing constant or time-varying toll and, as a true extension, the so-called “inverse Stackelberg game” with the road authority setting toll as a function of traffic flows in the network. In both situations the drivers are assumed to choose their routes in accordance with the dynamic logit-based stochastic user equilibrium. We formulate the dynamic optimal toll design problem with flow-dependent second-best tolling and present a solution algorithm. This algorithm will be applied in a small case study, where the tolls are affine functions of link traffic densities on tolled links. Even with this rather simple tolling one can improve the system performance remarkably. The results of flow-dependent tolling (Inverse Stackelberg) can never be worse than those of the tolling independent of traffic flow (Stackelberg). In some situations the optimal second-best flow-dependent toll can be decreasing with traffic flow. These phenomena will be discussed as well.

1 INTRODUCTION & LITERATURE OVERVIEW

Traffic congestion has become a big problem, especially in heavily populated metropolitan areas. With increasing occupancy of the road networks, the problem of congestion becomes more and more actual. Common methods used to alleviate congestion (such as improvements of junctions, broadcasting road conditions via traffic reports, building new roads and widening of existing ones, etc.) might be expensive, difficult to apply, and not very efficient. When it is not easy to apply the standard methods, traffic congestion can be reduced by imposing appropriate tolls, in a road pricing scheme. Road pricing has received a lot of attention in both research (e.g., (1), (2), (3), (4), (5)) and practice (6).

The idea of reducing congestion via appropriate tolls led to the introduction of the so-called *optimal toll design problem* ((2), (7)). This optimal toll design problem is a problem of the Stackelberg type ((8), (9)), applied to the traffic environment with a road authority as a leader and travelers as followers. The aim of the road authority is to minimize its objective function, which is dependent on the travelers' decisions, by choosing optimal tolls for a subset of links (so-called tollable links), while the travelers minimize their individual travel costs. Their behavior is usually modeled by applying a *traffic assignment* ((10), (11)). The traffic assignment (TA) consists of determining the routes used by network users, taking into account the link tolls.

There are extensive studies focusing on *static* optimal toll design problem, i.e., on problems, where decisions of the players (travelers and the road authority) do not evolve in time ((1), (2)). Although static models are still widely used, the theory and practice of *dynamic* models have evolved significantly over the last 10 years. In the dynamic version of the optimal toll design problem the *dynamic traffic assignment* (DTA) applies. DTA models typically describe route choice behavior of travelers on a transportation network and the way in which traffic dynamically propagates through the network.

If the travelers are assumed to have perfect information, the *deterministic user equilibrium* (DUE) applies (12), both in dynamic and static optimal toll design problem. Similarly, with imperfect information, the *stochastic user equilibrium* (SUE) is applied, for example a *logit-based stochastic equilibrium* (LB-SUE), see (13).

Considering possible tolling strategies there are two main research streams differing in the definition of the set of tollable links. With so-called first-best pricing all the links in the network can be tolled ((14), (15), (16)), with so-called second-best pricing not all links are tollable ((1), (17)). The latter concept is clearly more applicable in practice.

Dynamic congestion pricing models in which network conditions and link tolls are time varying, have been addressed in (18), where the effectiveness of various pricing policies (time-varying, uniform and step tolls) was compared as well. Only a bottleneck or a single destination-network is considered. Possibility of application of traffic-flow dependent tolls is not discussed here. In (19) and (20) dynamic marginal (first-best) cost pricing models for general transportation networks were developed. As indicated by the authors, the application of their model is limited to destination-specific (rather than route or link-based) tolling strategies, which might complicate its practical application. Moreover, only the first-best pricing is considered here.

Following extensive case studies of two-route congestion problem in static networks ((4), (21), (22)), we have introduced its (second-best) dynamic variant, where the link tolls are functions of link or route flows in the network, for only a proper subset of all links. This fits within the theoretical framework of “inverse Stackelberg” problems ((8),(23)). In (24) we have considered the dynamic optimal toll design problem with the road authority minimizing the total travel time of the system and travelers driven by DTA. We have solved the problem analytically for a 2-link network, also for the dynamic case. We have found out that the flow-dependent toll brings better results to road authority than the uniform or time-varying tolls.

This paper introduces an extension of our recent research to general networks. To the best of our knowledge, no research dealing with the dynamic optimal toll design problem with the second-best tolling, the travelers driven by LB-SUE, and the aim to find optimal toll defined as a function of the traffic flows in the network has been done before. Some authors (e.g., (18), (26), (27), (28)) consider the step-wise second-best tolling, though. In (7) the dynamic optimal toll design problem is considered with a case study on a very small network. Tolls are assumed uniform or time-varying, but traffic-flow-invariant, and the problem of finding the optimal toll is defined, but not solved here, albeit the impact of some specific toll values on travelers' route and departure time choices is presented. In (1) and (17) the static second-best tolling is considered, travelers are driven by deterministic user equilibrium (DUE), the objective function of the road authority is defined as the surplus of the network, the traffic demand is elastic, and it is assumed that the link cost functions are increasing with respect to traffic flows. In (16) a very general static Stackelberg model is presented, where the road authority has two decision variables, one of them possibly traffic-flow-dependent. The paper itself deals with general mathematical properties of traffic equilibria, however. The tolls are assumed to be constant and the traffic-flow-dependent variable is interpreted as management decisions of the road authority.

The contributions of the paper can be listed as follows:

- The dynamic second-best optimal toll design problem with the traffic flow-dependent tolling is formulated. Up to now flow-dependent tolling appeared in static networks, in heuristic studies, or with the first-best pricing only.
- A simple algorithm solving the problem is introduced and presented on a small case study. This algorithm is applicable to general networks.
- We show that the flow-dependent tolling can never be worse than the flow-independent one.
- Phenomena of optimal toll decreasing with traffic flow is presented and discussed.
- Introduction of second-best flow-dependent tolls in dynamic optimal toll design problem is very promising for practical applications. This topic is discussed here as well.

Some simplifying assumptions are made in this paper. For example, the travelers are assumed to be uniform, and they make route choices, but do not choose their departure times. Models with flow-invariant tolling, where the travelers choose their departure times as well, are discussed in e.g. (7), (18). The currently used model is sufficient for our research purposes and can be extended after the role of flow-dependent toll is elaborated in more detail.

This paper is composed as follows: In Section 2 we will introduce a basic concept of Stackelberg and inverse Stackelberg games, mainly by means of examples. In Section 3 the optimal toll design problem is presented, including some properties and examples of interesting phenomena, which could be explained analytically. In Section 4 the algorithm used for solving the problem is described. In Section 5 we show the case studies on a small network. The outcome of the Stackelberg game is compared with that of the inverse Stackelberg game. Conclusions and future research are discussed in Section 6.

2 STACKELBERG AND INVERSE STACKELBERG GAMES

Let us consider two players, which we call leader and follower. The leader has the decision variable $u_L \in D_L \subset \Re$, while the follower has the decision variable $u_F \in D_F \subset \Re,$, where D_L and D_F are the decision spaces of the leader and the follower, respectively, and $\Re$ is the set of real numbers. The leader and the follower have the real-valued cost functions $J_L(u_L, u_F)$, $J_F(u_L, u_F)$, respectively. We assume that the cost functions as well as D_L and D_F are known to both players.

Each player chooses his own decision variable in such a way as to minimize his own cost function. Some well-known equilibrium concepts, such as the Stackelberg equilibrium concept (see (8)), can be used to define a solution. With the Stackelberg equilibrium concept the leader announces his decision u_L, which is subsequently made known to the other player, the follower. With this knowledge the follower picks u_F. Hence, u_F becomes a function of u_L, written as $u_F = \rho_F(u_L)$, which is determined through the relation

$$\min_{u_F} J_F(u_L, u_F) = J_F(u_L, \rho_F(u_L)).$$

It is assumed that this minimum exists and that it is unique for each possible choice u_L by the leader. The function $\rho_F(\cdot)$ is sometimes called a reaction function (it indicates how the follower will react upon the leader's decision). Before the leader announces his decision u_L he will realize how the follower will react; hence, the leader will choose, and subsequently announce u_L so as to minimize $J_L(u_L, \rho_F(u_L))$.

Example 1.

Suppose

$$J_L(u_L, u_F) = u_L^2 + (u_F - 5)^2, \quad J_F(u_L, u_F) = u_L^2 + u_F^2 - u_L u_F .$$

Trivially, the reaction curve is $u_F = \rho_F(u_L) = \frac{1}{2} u_L$. Hence, the leader will choose u_L so as to minimize $J_L\left(u_L, \frac{u_L}{2}\right)$, which immediately results in $u_L = 2$. With this decision of the leader, the follower will choose $u_F = 1$. The costs for the leader and the follower are given by 20 and 3, respectively.
In Figure 1 the important contours of the leader's and the follower's cost functions and the reaction curve of the follower are depicted. Point $(u_L, u_F) = (2,1)$ is the closest point to the point $(u_L, u_F) = (5,0)$ (minimum of $J_L(u_L, u_F)$) that lies on the reaction curve of the follower.

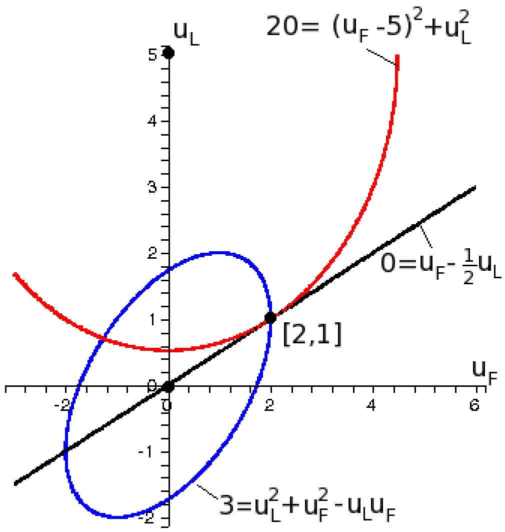


FIGURE 1 Graphical illustration of Example 1

□

Another equilibrium concept, to be dealt with now, is the inverse Stackelberg equilibrium discussed in, e.g., (23) and (24). The leader does not announce the scalar u_L, as above, but a “decision rule” given by the function $\rho_L(\cdot): D_F \to D_L$.

Examples of games of this type:

- The leader is the government and the follower is a citizen. The government demands from the citizen taxes dependent on the income u_F of the citizen. It is up to the citizen as to how much money he will earn. The income tax the government will receive equals $\rho_L(u_F)$, where the “rule for taxation” $\rho_L(\cdot)$ was made known ahead of time.
- The leader is a producer of electricity in a liberalizing market and the follower is the market (a group of clients) itself. The price of electricity is set to $\rho_L(u_F)$, where u_F is the amount of electricity traded.
- The leader is a road authority and the followers are travelers on the road network. Travelers make their travel decisions so as to minimize their travel costs. These travel costs consist of a travel-time dependent part and link tolls, to be paid by travelers using tolled links and imposed by the road authority as functions of the traffic flows in the network, u_F. The road authority minimizes the total travel time of the network by setting tolls on tollable links. These tolls $\rho_L(u_F)$, defined as mappings of the link traffic flows in the network, are announced in advance.

This problem is the main subject of this paper and will be discussed from Section 3. Since this game involves many followers, an additional item to be discussed is how the followers will react to each others' decisions.

To recapitulate, in a Stackelberg setting, the leader announces and acts first and subsequently acts the follower. In an inverse Stackelberg setting, however, while the leader announces his “decision rule”, the follower acts first and the leader second.

Given the function $\rho_L(\cdot)$ the follower (in this section we assume one follower) will make his optimal choice u_F^* according to

$$u_F^* = \arg\min_{u_F} J_F(\rho_L(u_F), u_F). \tag{2}$$

The leader, before announcing his $\rho_L(\cdot)$, will realize how the follower will play and he can exploit this knowledge in order to choose the best possible ρ_L-function, such that ultimately his own cost function J_L is minimized. Symbolically, we could write

$$\rho_L^*(\cdot) = \arg\min_{\rho_L(\cdot)} J_L\left(\rho_L\left(u_F^*(\rho_L(\cdot))\right), u_F^*(\rho_L(\cdot))\right). \tag{2}$$

In this way one enters the realm of composed functions (see 30) which is known to be a notoriously complex area. From here onward it turns out to be difficult to proceed in an analytic way. However, there is a trick that sometimes works, as shown in the following example.

Example 2.

Suppose that the cost functions are those of Example 1. If both the leader and the follower would be so kind and minimize $J_L(u_L, u_F)$, the follower totally disregarding his own cost function, the leader would obtain

$$\min_{u_L, u_F} J_L(u_L, u_F) = J_L(0,5) = 0.$$

This value is called the team minimum. Now the leader should choose the curve $u_L = \rho_L(\cdot)$ in such a way that the point $(u_L, u_F) = (5,0)$ lies on this curve and, moreover, that this curve does not have other points in common with the set

$$\{(u_L, u_F) : J_F(u_L, u_F) \le J_F(0,5)\}.$$

To keep the problem simple a linear curve satisfying this property is chosen. Clearly, there exists only one line satisfying the required conditions and it is line

$$u_L = \rho_L(u_F) = 2u_F - 10.$$

With this choice of the leader, the best for the follower to do is to minimize

$$J_F(2u_F - 10, u_F),$$

which leads to $u_F = 5$. The situation is depicted in Figure 2. Hence $u_L = 0$ and, interestingly, the leader obtained his team minimum in spite of the fact that the follower minimized his own cost function (though with the constraint $u_L = \rho_L(u_F) = 2u_F - 10$. The costs for the leader and the follower will be 0 and 25, respectively. Compared with Example 1, the leader is much better off ($0 < 20$) while the follower is worse off ($25 > 3$).

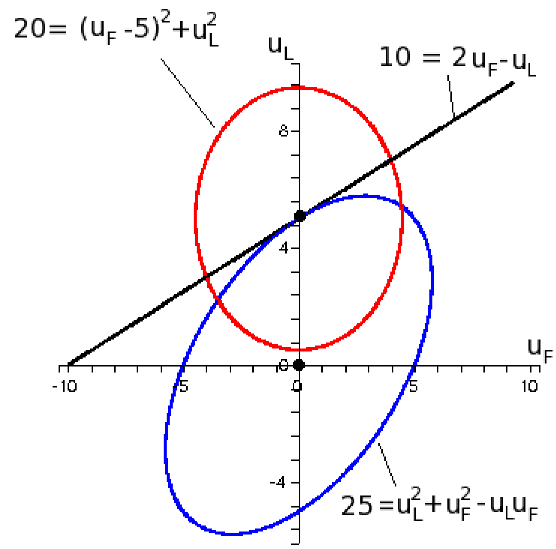


FIGURE 2 Graphical illustration of Example 2

□

In this notion, a "classical" Stackelberg game is a special case of an inverse Stackelberg game with $\rho_L(\cdot)$ chosen as a constant value.

Other examples exist in which the leader cannot obtain his team minimum, and such problems are harder to deal with, but the advantage of use of the inverse Stackelberg strategy will stay the same as in our example.

In the following sections we will formulate the Dynamic optimal toll design problem as an inverses Stackelberg game with tolls dependent on traffic flows.

3 DYNAMIC OPTIMAL TOLL DESIGN PROBLEM

3.1 Preliminaries

Let $G=(N,A)$ be a road network, where N and A are finite nonempty sets of nodes and directed arcs (links), respectively. Let $T \subset A$ be a set of tollable links, initially given. Let K ($|K| \in \aleph$) be the set of time instances. The set of tollable links will be denoted by $T \subset A$. There is a set of origin-destination pairs $RS \subset N \times N$. For all ordered pairs of nodes $(r,s) \in RS$, where r is an origin, s is a destination, and for each time instance $k \in K$, there is a positive travel demand on travelers $d^{(r,s),k}$ [veh]. The network is assumed to be strongly connected, that is, at least one route connects each (r,s) - pair. Let P be the set of all simple paths (i.e., paths without cycles) in the network and let $P^{(r,s)} \subset P$ be the set of all paths (routes) between origin-destination pair (r,s). An element of P will be denoted by p, an element of $P^{(r,s)}$ will be denoted by $p^{(r,s)}$. The travel cost on route p, respective $p^{(r,s)}$, as experienced by an individual user entering this route at time instance k, will be referred to as to c_p^k and $c_p^{(r,s),k}$, respectively, similarly the travel times at time instance k will be denoted by τ_p^k and $\tau_p^{(r,s),k}$, respectively. The dynamic route flow rate on the route $p^{(r,s)}$ at time instance k will be denoted by $f_p^{(r,s),k}$, similarly the dynamic route flow rate on the route p at time instance k will be denoted by f_p^k.

For each link $a \in A$ in network G the following parameters are initially given: link length s_a [km], maximum speed $\vartheta_a^{\max}$ [km/h], minimum speed $\vartheta_a^{\min}$ [km/h], critical speed ϑ_a^{crit} [km/h], jam density J_a^{jam} [pcu/km], and the unrestricted link capacity C_a [pcu/h]. The link travel time τ_a^k [h] on link $a \in A$ for an individual user entering this link at time instance k can be computed as

$$\tau_a^k = \frac{s_a}{\vartheta_a^k}, \tag{1}$$

where ϑ_a^k is a speed of travelers entering link a at time instance k, defined by the Smudlers speed-density function:

$$\vartheta_a^k = \begin{cases} \vartheta_a^{\max} + \dfrac{\vartheta_a^{crit} - \vartheta_a^{\max}}{J_a^{crit}} J_a^k, & \text{if } J_a^k \le J_a^{crit}, \\ J_a^{jam} + \left(\vartheta_a^{crit} - \vartheta_a^{\min}\right) \dfrac{\dfrac{1}{J_a^k} - \dfrac{1}{J_a^{jam}}}{\dfrac{1}{J_a^{crit}} - \dfrac{1}{J_a^{jam}}}, & \text{if } J_a^{crit} \le J_a^k \le J_a^{jam}, \\ \vartheta_a^{\min}, & \text{if } J_a^k \ge J_a^{jam}. \end{cases} \tag{2}$$

The critical density J_a^{crit} can be computed as $J_a^{crit} = \dfrac{C_a}{\vartheta_a^{crit}}$.

The dynamic link cost on link a as experienced by a single traveller entering this link at time instance k is defined as

$$c_a^k = \alpha \tau_a^k + \theta_a^k, \tag{3}$$

where α is travelers' value of time [veh/h] and θ_a^k is the link toll set by the road authority on link a at time instance k. The dynamic route travel time τ_p^k [h] is assumed to be additive, i.e.,

$$\tau_p^k = \sum_{a\in A}\sum_{k'\in K}\delta_{a,p}^{k,k'}\cdot\tau_a^{k'}, \qquad (4)$$

where $\delta_{a,p}^{k,k'}$ is a *dynamic route-link incidence indicator* defined as

$$\delta_{a,p}^{k,k'} = \begin{cases} 1, & \text{if travelers departing at time inst. } k \text{ reach link } a \text{ at time inst. } k', \\ 0, & \text{otherwise.} \end{cases} \qquad (5)$$

Similarly, we assume that the dynamic route tolls and the dynamic route costs are additive, i.e.,

$$\theta_p^k = \sum_{a\in A}\sum_{k'\in K}\delta_{a,p}^{k,k'}\cdot\theta_a^{k'}, \qquad (6)$$

$$c_p^k = \sum_{a\in A}\sum_{k'\in K}\delta_{a,p}^{k,k'}\cdot c_a^{k'}. \qquad (7)$$

On the other hand, the dynamic link flow rates q_a^k are additive with respect to dynamic route flow rates, i.e.,

$$q_a^k = \sum_{a\in A}\sum_{k'\in K}\delta_{a,p}^{k,k'}\cdot f_p^{k'}. \qquad (8)$$

The route flow rates have to satisfy the flow-conservation constraint given as

$$\sum_{p^{(r,s)}\in P^{(r,s)}} f_p^{(r,s),k} = d^{(r,s),k}, \quad \forall (r,s)\in RS, \qquad (9)$$

as well as the non-negativity constraint given by

$$f_p^{(r,s),k} \ge 0, \quad \forall k\in K,\ \forall p^{(r,s)}\in P^{(r,s)},\ \forall (r,s)\in RS. \qquad (10)$$

The set Q^k defined as

$$Q^k \overset{def}{=} \left\{ \left(f_1^k,\ldots,f_{|P|}^k\right): \sum_{p^{(r,s)}\in P^{(r,s)}} f_p^{(r,s),k} = d^{(r,s),k}, \quad f_p^{(r,s),k}\ge 0, \quad \forall p^{(r,s)}\in P^{(r,s)}, \forall (r,s)\in RS \right\} \qquad (11)$$

will be called the *set of feasible route flow rates at time instance* k. Here $Q^k = Q_1^k \times \ldots \times Q_{|P|}^k \subset \left(\Re_+^0\right)^{|P|}$, where for each feasible route flow on route p at time instance k it follows that $f_p^k \in Q_p^k$.

The link dynamics is defined by the *Dynamic network loading* (DNL) model. The DNL model is formulated as a system of equations e*xpressing link dynamics*, *flow conservation*, *flow propagation* and *boundary constraints*. This DNL model is adopted from (28) and is not further discussed here.

3.2 Travelers' behavior

Travelers entering the network at each time instance k minimize their perceived travel costs. We assume that in equilibrium state, any traveler at any time instance cannot minimize his perceived travel costs by unilateral change of his route. This situation fits within the frame of dynamic stochastic user equilibrium model (See (29)), which can be written as

$$f_p^{(r,s),k} = \Psi_p^{(r,s),k}\cdot d^{(r,s),k}, \quad \forall p\in(r,s), \quad \forall (r,s)\in RS, \qquad (12)$$

where $\Psi_p^{(r,s),k}\in[0,1]$ determines the probability that route $p^{(r,s)}$ is perceived as the cheapest, given actual travel times (see (10)) and is dependent on the vector of route costs between the (r,s)-pair at time instance k. In this paper the dynamic *logit-based stochastic user equilibrium* (LB-SUE) in which $\Psi_p^{(r,s),k}$ is defined as

$$\Psi_p^{(r,s),k} = \frac{\exp\left(-\mu c_p^{(r,s),k}\right)}{\sum_{p' \in P^{(r,s)}} \exp\left(-\mu c_{p'}^{(r,s),k'}\right)}, \tag{13}$$

is used. In (7) the parameter μ is a positive parameter associated with the random cost component. If the value of μ is large, the perception error is small, and travelers will tend to choose minimum-cost routes. The costs perception tends towards being accurate as $\mu \to \infty$, which then simplifies to the dynamic deterministic route choice equilibrium assignment (See (29)).

3.3 The aim of the road authority

In the inverse Stackelberg setting the road authority sets tolls as mappings of traffic flows in the network. We assume that these tolls are twice continuously differentiable functions of the link flows. The road authority sets at each time instance vector of the link flows $\Theta^k(\cdot) = (\theta_1^k(\cdot), \ldots, \theta_{|A|}^k(\cdot))$ in such a way as to minimize the total travel time of the system. Each $\theta_a^k(\cdot)$ is a function from Q^k into $\Re_+^0$. The sequence of all vectors $\Theta^k(\cdot)$ will be denoted by $\Theta(\cdot)$, i.e., $\Theta(\cdot) = \left(\Theta^1(\cdot), \ldots, \Theta^{|K|}(\cdot)\right)$. Moreover,

$$\theta_a^k\left(q_a^k\right) \geq 0 \quad \forall q_a^k \in Q_a^k,\ \forall a \in T,\ \forall k \in K, \qquad \theta_a^k\left(q_a^k\right) = 0 \quad \forall a \in A \setminus T, \forall k \in K. \tag{14}$$

With travelers driven by LB-SUE the problem boils down to finding a vector of continuously differentiable functions $\Theta^*(\cdot)$ such that

(P)
$$\Theta^*(\cdot) = \arg\min_{\Theta(\cdot)} \sum_{k \in K} \sum_{(r,s) \in RS} \sum_{p \in P^{(r,s)}} \tau_p^{(r,s),k} \cdot f_p^{(r,s),k}, \tag{15}$$

subject to (1)-(4), (7)-(10),(14), and

$$f_p^{(r,s),k} = \frac{\exp\left(-\mu c_p^{(r,s),k}\right)}{\sum_{p' \in P^{(r,s)}} \exp\left(-\mu c_{p'}^{(r,s),k'}\right)} \cdot d^{(r,s),k}, \quad \forall p \in (r,s), \quad \forall (r,s) \in RS. \tag{16}$$

Clearly, the uniform or time-varying problem is a specific case of (P). In that situation the tolling functions $\theta_a^k(\cdot)$ are defined as constants, i.e., for each k there would be a nonnegative constant β^k such that

$$\theta_a^k\left(q_a^k\right) \stackrel{def}{=} \beta^k \in \Re_0^+. \tag{17}$$

This means that the Stackelberg game within the framework dynamic optimal toll design problem is a special case of the inverse Stackelberg game within the same framework (with the same initial conditions). In other words, the result of the Stackelberg game can never be better than that of the inverse Stackelbrg game, provided that the initial conditions are the same for both games. This result is not surprising since it follows from Section 2.

3.4 The problem

We are looking for the solution of problem (P). The comparison of the traffic-flow dependent tolling outcome with outcomes of traffic-flow-invariant tolling and no tolling is also subject of our research. Brief discussion about the properties of the total travel time function will take place as well.

4 SOLUTION APPROACH

In this section we will present an algorithm to solve the problem. Network $G(N,A)$, set K, and travel demands $d^{(r,s)}(k)$ are initially given, as well as the set of routes P, $\varepsilon_{\max}$ ($0<\varepsilon_{\max}<<1$), the set of tollable links $T\subset A$, and a finite set Ω of possible tolling functions, where $\Omega=\bigcup_{k\in K}\Omega^k$.

The algorithm has built-in two optimization procedures: *outer loop* and *inner loop*.

In the *outer loop* the road authority minimizes the total travel time of the system by setting tolling functions on the tollable links. The tolling function from Ω that is minimizing the total travel time of the system, is taken as an optimal strategy for the road authority.

In the *inner loop* the *dynamic route choice model,* aiming to determine a stochastic dynamic user-equilibrium based on the actual route travel costs, is applied. In each iteration, new route flow proportions over route set P^{rs} are computed using a dynamic logit model. This gives new route flow rates $f^{rs}(k)$ that are passed on to the dynamic network loading (DNL) model. In order to speed up convergence, the *method of successive averages* (MSA) is adopted on the route flow level (See (10).):

$$f_p^{(r,s),k,(i)}=f_p^{(r,s),k,(i-1)}+\delta\left(\Psi_p^{(r,s),k,(i)}d^{(r,s),k}-f_p^{(r,s),k,(i-1)}\right), \tag{18}$$

where steplength δ is set to $i^{-2/3}$, $f_p^{(r,s),k,(i)}$ is the route flow rate on route $p^{(r,s)}$ at time instance k for the *i*-th iteration, $\Psi_p^{(r,s),k,(i)}$ is the probability that the route $p^{(r,s)}$ is perceived as the cheapest at time instance k the *i*-th iteration, $d^{(r,s),k}$ is the travel demand for the (r,s)-pair at time instance k. Formula (13) is used for its computation.

The *dynamic network loading (DNL) model* simulates the route flows $f^{(r,s)}(k)$ along the links in the network. This model is at the heart of the DTA model and is also the most computationally intensive part. The convergence criterion of the DTA is reached using the *relative dynamic duality gap function* $\varepsilon^{(i)}$ defined for the *i*-th iteration as:

$$\varepsilon^{(i)}=\frac{\sum_{(r,s)\in RS}\sum_{p^{(r,s)}\in P^{(r,s)}}\left(c_p^{(r,s),k,(i)}-\pi^{(r,s),k,(i)}\right)f_p^{(r,s),k,(i)}}{\sum_{(r,s)\in RS}\pi^{(r,s),k,(i)}\cdot d^{(r,s)}}. \tag{19}$$

Here $\pi^{(r,s),k,(i)}$ is a minimal route travel time between the (r,s)-pair at time instance k for the *i*-th iteration, $c_p^{(r,s),k,(i)}$ is the cost on route $p^{(r,s)}$ at time instance k for the *i*-th iteration, similarly $f_p^{(r,s),k,(i)}$ is the route flow rate on route $p^{(r,s)}$ at time instance k for the *i*-th iteration.

If the duality gaps of two consecutive iterations are close enough, i.e., if $\left|\varepsilon^{(i)}-\varepsilon^{(i-1)}\right|<\varepsilon_{\max}$, the algorithm is terminated. Here $\varepsilon_{\max}$ is a positive number close to zero defined beforehand. The scheme of the algorithm is depicted in Figure 3.

Remarks:

- The algorithm is trivially convergent, if the set Ω is finite. If the set Ω is infinite, the problem is NP-hard. The convergence of the inner loop of the algorithm was shown in, e.g., (29).
- The solution of the problem is generally non-unique, which trivially follows from the problem's definition. The uniqueness would be satisfied only if the link and route costs would be strictly increasing with the traffic flow rates.
- The outer level of the algorithm has to be specified with respect to the properties of Ω .

Step 1: Initialization
Download $G(N,A)$, define K, RS, T, $d^{(r,s)}(k)$, $P^{(r,s)}$, Ω for each k and $(r,s)\in RS$. Define $\varepsilon_{\max}$, μ, define $TTT_{\min}=\infty$; Set network empty.
Step 2: Outer loop
  i=0; $\varepsilon_0=\infty$;
  for every $\Theta(\cdot)\in\Omega$ do
    i=i+1;
    Step 3: Inner loop (DTA)
      while $\left|\varepsilon^{|i|}-\varepsilon^{|i-1|}\right|\geq\varepsilon_{\max}$ *do for all* $k\in K$
        Compute dynamic link costs from (3) and dynamic route costs from (7);
        Determine the route choices of travellers for every k *from (16);*
        Update dynamic route flows using (18);
        Perform DNL to obtain link flows;
      end do;
      Compute the total travel time function TTT corresponding to $\Theta(\cdot)\in\Omega$ *;*
      if $TTT<TTT_{\min}$
        TTT_{min} *:=TTT;*
      end if;
  end do;
  Return TTT_{min}, $\Theta^*(\cdot)$.

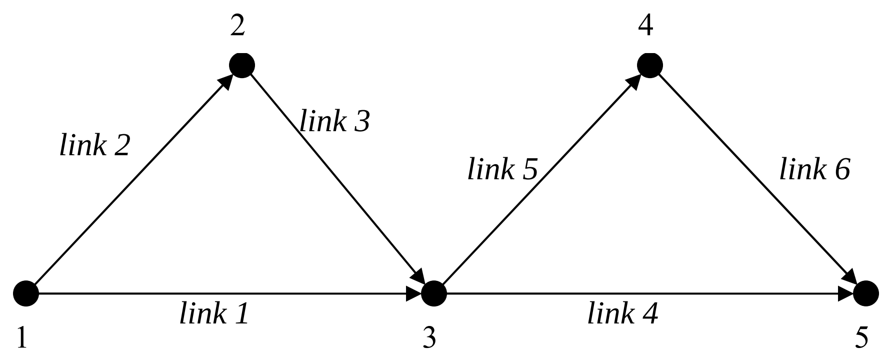
FIGURE 3 Scheme of the algorithm

5 CASE STUDY

In this section the algorithm introduced in Section 4 will be applied on Chen network, which is depicted in Figure 2. The link parameters are defined in Table 2. We consider 23 departure time intervals, the demand step is 12 minutes. Also, each demand step is divided into 10 substeps splitting each interval into 10 equal subintervals, ξ will refer to the ξ-th subinterval, $\xi\in\{1,\ldots,230\}$. The travel demands are defined in Table 2.

link	length [km]	maximum speed [km/h]	critical speed [km/h]	minimum speed [km/h]	jam density [pcu/km]	capacity [pcu/h]
1	10	120	80	10	150	2000
2	7.5	120	80	10	150	2000
3	7.5	120	80	10	150	2000
4	10	120	80	10	150	2000
5	7.5	120	80	10	150	2000
6	7.5	120	80	10	150	2000

TABLE 1 Link parameters



RS – pairs: $\{(1,5), (3,5)\}$

Tollable links: $T = \{1,4\}$.

RS-pair	route	links
(1,5)	$p_1^{(1,5)}$	1,4
(1,5)	$p_2^{(1,5)}$	2,3,4
(1,5)	$p_3^{(1,5)}$	1,5,6
(1,5)	$p_4^{(1,5)}$	2,3,5,6
(3,5)	$p_1^{(3,5)}$	4
(3,5)	$p_2^{(3,5)}$	5,6

FIGURE 3 CHEN network

k	1,20,21,22,23	2, 19	3, 18	4, 17	5, 16	6, 15	7,14	8,13	9,10,11,12
$d^{(1,5),k}$	0	300	600	1500	3000	4500	7500	10500	12000
$d^{(1,5),k}$	0	200	400	1000	2000	3000	5000	7000	8000

TABLE 2 Travel demands

We will restrict ourselves by defining the tolls on tollable links as linear functions of traffic volumes in the network in the following manner:

$$\theta_a^\xi\left(x_a^\xi\right) \stackrel{def}{=} \begin{cases} a\dfrac{x_a^\xi}{s_a}+b, & \text{if } a\in T \text{ and } x_a^\xi>0, \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

Here x_a^ξ is the link volume in the ξ-th subinterval, i.e., the number of travellers present on link a during the ξ-th time subinterval. Since the toll functions have properties defined by (12), the outer loop of the algorithm presented in Section 5 simplifies to finding a and b minimizing the total travel time of the system. Moreover, if a is set to 0, we obtain classical Stackelberg problem with traffic flow–invariant tolls.

The grid search is applied for the outer loop, since we are interested in the shape of the objective function as well. In the future research some more sophisticated method taking into account non-linearity and non-convexity of this total travel time function can be used (e.g. genetic algorithms). An upper bound on link toll is added: $\theta_a \le \theta_{\max}$, $\forall a \in A,$ where $\theta_{\max}$ was set to 20 euro. The other initial parameters are defined as follows: $\mu = 0.2,$ $\varepsilon_{\max} = 10^{-5}$.

Obtained results with no toll, uniform toll, and flow-varying toll are depicted in Table 3, while the interpolation of the total travel time function with respect to different values of a and b (on the region where the global minimum was found) is presented in Figure 4. The function itself is obviously non-linear and non-convex, but smooth (the nonsmoothness in the plot is caused by the grid search).

	Minimum total travel time [h]	Optimal toll scheme
No toll	19021.2	-
Uniform toll – link 1 (SG)	18868.9	$\theta_1^* = 20$ [euro]
Uniform toll – both links $1,4$ the same toll (SG)	18313.6	$\theta_1^* = 18.5, \theta_4^* = 18.5$ [euro]
Uniform toll –links $1,4$ different toll (SG)	18101.6	$\theta_1^* = 19.4, \theta_4^* = 10.3$ [euro]
Flow-dependent toll on both $1,4$ (ISG)	16619.7	$a^* = -4.2, b^* = 1.8$

TABLE 3 Results

As you can see, the flow-dependent tolling brings us remarkable improvement of the total travel time. The optimal tolling functions are on both links given by

$$\theta_a^\xi\left(x_a^\xi\right) = 1.8 - 4.2\frac{x_a^\xi}{s_a}. \tag{21}$$

The resulting tolls on links 1 and 4 are then between 0 and 1.5 euro for each time subinterval.

Remark

One way of decreasing number of travelers on the route is to increase toll on that route. The other option is to decrease toll on its route alternative, as it was heuristically done in San Diego's interstate 15 congestion pricing project (see (6)). That is also why the optimal tolling function (21) is decreasing with traffic volume. This is interesting mainly from practical point of view, since change of the network properties do not need to urge a change of a set of tollable links, recalculation of the existic tolling scheme might be sufficient.

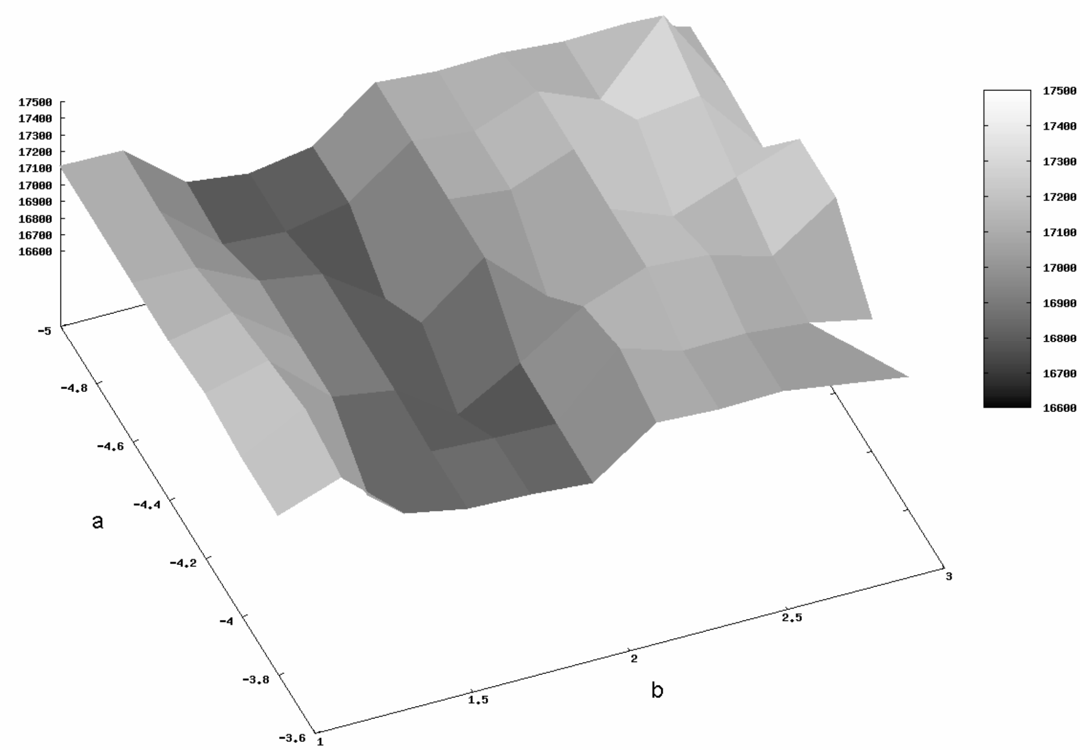


FIGURE 4 Total travel time function – inverse Stackelberg game

6 CONCLUSIONS, FUTURE RESEARCH

We formulated the dynamic optimal toll design problem with second-best traffic-flow dependent tolling as an inverse Stackelberg game. We discussed the properties of the model and presented a simple algorithm solving the problem. This algorithm was performed on a small case study. Even with the very simple tolling schemas, application of the traffic-flow dependent tolling improved the system performance remarkably.

The resulting total travel function is smooth, but non-linear and non-convex, the problem itself is NP-hard, if we do not restrict ourselves on finite set of possible tolling functions.

An extension of the presented model to multi-class user conditions, using of more advanced tolling rules, as well as applying the presented algorithm on bigger networks is a subject of future research. In that case, a more sophisticated procedure solving the outer loop of the problem has to be introduced. We are currently working on solving the problem on real-size network with use of genetic algorithms that seem to be a promising tool for this purpose.

Clearly, the traffic-flow dependent second-best tolling is a very promising pricing option for real applications, since it brings generally better results than uniform or time-varying traffic-flow independent tolling.

One of the important questions concerning the traffic-flow dependent tolling is how to apply our approach in practice. In San Diego's interstate 15 congestion pricing project (see (6)) the toll on one route is automatically decreased when its parallel route is congested. Similar concept can be used when applying the traffic-flow dependent tolling. Moreover, the travelers could be informed about the actual level of toll via information boards before they enter the route. Additional research is needed in this area, too.

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