On Dynamic Optimal Toll Design Problem with Traffic-flow Dependent Tolls and Drivers’ Joint Route and Departure Time Choices

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Abstract—In this paper the dynamic optimal toll design problem as a game of the Stackelberg type is investigated, with the road authority as leader and drivers on the road network as followers. The road authority sets dynamic traffic-flow dependent tolls on some links in order to minimize its objective function, while the drivers choose their routes and departure times so as to minimize their own perceived travel costs, which include a travel time component, tolls, and penalties for deviation from their preferred arrival and departure times. The drivers’ behavior is modeled using a dynamic user equilibrium model. We define the problem in a general form so that a wide class of objective functions for the road authority and a wide class of the dynamic traffic equilibria can be employed. We also discuss the problem properties. The problem to find optimal traffic-flow dependent terrors in a general setting introduced in this paper is NP-hard. One of the ways of tackling such problems is to use advanced heuristic methods. In this paper a neurosimulation-based approach is proposed, using the neurosimulator FAUN 1.0.

The proposed solution method is illustrated on case studies with the so-called Chen network.

I. INTRODUCTION & LITERATURE OVERVIEW

Traffic congestion has become a big problem, especially in heavily populated metropolitan areas. One of the ways to reduce traffic congestion is to impose appropriate tolls, in a road pricing scheme. Road pricing has received a lot of attention in research ([1], [2], [3], [4]) and practice ([5]).

The dynamic optimal toll design problem, discussed in this paper ([2], [6], [7]) is a problem of the Stackelberg type ([8], [9]), applied to the traffic environment with a road authority as a leader and drivers (also referred to as travelers or users) as followers. The aim is to minimize the objective function of the road authority by imposing appropriate tolls on some links, while the drivers minimize their perceived travel costs by their travel choices, taking the tolls into account. In an equilibrium state, a dynamic user equilibrium applies ([10], [11]).

A number of studies have focused on the so-called first-best pricing, in which all the links in the network can be tolled ([12], [13], [14]). In our research we consider the so-called second-best pricing ([1], [15]), in which only a proper subset of all links can be tolled, as this concept seems to be more suitable for practical applications.

Most publications dealing with the optimal toll design problem with second-best pricing ([6], [1]) aim to find the optimal link tolls as values independent of the traffic flows in the network (uniform or time-varying toll). We have recently introduced ([16], [17]) the optimal toll design problem, in which the link tolls are computed as functions of link or route traffic flows in the network. This fits within the theoretical framework of the so-called inverse Stackelberg problems ([18], [19]). We have shown that the traffic-flow dependent toll can improve the traffic system performance remarkably, while it can never perform worse than the traffic-flow invariant toll.

In our previous work we have assumed that travelers decide only about their routes, while their departure times are fixed. This paper introduces an extension of our recent research into the situation in which the drivers choose their departure times, too. Moreover, we generalize the problem so that a wide range of objective functions for the road authority and user equilibrium models for the drivers can be considered. We present some of the properties of this generalized problem. A neural-networks based optimization algorithm is used to solve the problem, as the problem is NP-hard. The proposed algorithm is can be used for general networks and for a wide range of pricing problems. The performance of the algorithm is presented on the so-called Chen network ([20]).

This paper is organized as follows: In Section II, the optimal toll design problem is defined and its properties are discussed. In Section III, the algorithm that is used for solving the problem is presented. Its advantages and pitfalls are discussed. In Section IV we perform experiments on the Chen network. Conclusions and possible future research are given in Section V.

II. THE DYNAMIC OPTIMAL TOLL DESIGN PROBLEM

A. Preliminaries

Let \( G = (\mathcal{N}, \mathcal{A}) \) be a strongly connected road network, where \( \mathcal{N} \) and \( \mathcal{A} \) are finite nonempty sets of nodes and directed links, respectively. Let \( \mathcal{P} \subseteq \mathcal{A} \) be the set of tollable links. There is a finite, nonempty set of origin-destination pairs \( \mathcal{R} \subset \mathcal{N} \times \mathcal{N} \). Let \( \mathcal{K} = \{1, 2, \ldots, |\mathcal{K}|\} \) be a time index set. Here each \( k \in \mathcal{K} \) refers to

- the interval \([k - 1.5\Delta, k - 0.5\Delta]\) if \( k \geq 2 \),
- the interval \([0, 0.5\Delta]\) if \( k = 1 \),

where \( \Delta \) [h] is the length of each time interval.

For an ordered pair of nodes \((r, s) \in \mathcal{R} \) where \( r \) is an origin and \( s \) is a destination, there is a positive number of drivers traveling from \( r \) to \( s \), the so-called travel demand \( d^{(r,s)}[\text{veh}]\), during the time interval \([0, |\mathcal{K}| - 0.5\Delta]\).

Let \( \mathcal{P} \) be the set of all simple paths in the network and let \( \mathcal{P}^{(r,s)} \subset \mathcal{P} \) be the set of simple paths between from \( r \) to \( s \). Each path is formed by one or more directed links. The
route flow on path \( p \in \mathcal{P} \) during the \( k \)-th time interval will be denoted by \( f_p^k \) (veh/h), the link flow on the link \( a \) during the \( k \)-th time interval will be denoted by \( q_a^k \) (veh/h).

Let \( \xi_p^k \) (euro) denote the route travel costs on the route \( p \in \mathcal{P} \) for a driver entering this route during the \( k \)-th time interval. Let \( \bar{q}_a^k \) (euro) denote the average link travel cost on a link \( a \) during the \( k \)-th time interval.

The traffic, times, costs, flows, and dynamic tolls are assumed to be additive. See [18] for a detailed description via the so-called dynamic route-arc incidence indicator.\(^1\)

For each route \( p \in \mathcal{P}(r,s) \) the dynamic route travel cost \( c_p^k \) for the travelers entering the route during the \( k \)-th time interval is defined as follows:

\[
c_p^k = \alpha_1 t_p^k + \theta_p^k + \alpha_2 (k - k_{(r,s),dep}) + \alpha_3 (k + \tau_p^k - k_{(r,s),arr}).
\]

where \( t_p^k \) is the actual route travel time, \( \theta_p^k \) is the actual route toll, \( k_{(r,s),dep} \) is the preferred departure time for a driver traveling from origin \( r \) to destination \( s \), starting his/her trip during the \( k \)-th time interval, \( k_{(r,s),arr} \) is the preferred arrival time for a driver traveling from origin \( r \) to destination \( s \), starting his/her trip during the \( k \)-th time interval; coefficients \( \alpha_1 \), \( \alpha_2 \), and \( \alpha_3 \) (euro/h) are the value of time, the penalty for deviating from the preferred departure time, and the penalty for deviating from the preferred arrival time, respectively.

Let \( q, t, \) and \( \xi \) denote the column vectors of the link flows, the link travel times, and the link travel costs for all links and all time intervals and let \( f, \tau, \) and \( c \) denote the column vectors of the route flows, the route travel times, and the route travel costs for all link and all time intervals (For more details, see [18]).

For each link from \( \mathcal{P} \) and each time interval a traffic-flow dependent toll can be imposed. The traffic-flow dependent toll on link \( a \in \mathcal{P} \) will be denoted by \( \theta_a^k(.) \). This toll will be defined for each \( k \)-th time interval as a polynomial function of link flow for the same time interval and on the same link, i.e.,

\[
\theta_a^k(q_a^k) = \sum_{m=0}^{M} w_{a(m),k} (q_a^k)^m,
\]

where \( w_{a(m),k} \) is defined as follows:

\[
w_{a(m),k} = \begin{cases} 0 & \text{for } a \in \mathcal{P} \setminus \mathcal{P} \text{ for } a \in \mathcal{P}, \\ \in \mathbb{R} & \text{for } a \in \mathcal{P}. \end{cases}
\]

with \( M \in \mathbb{N}_0 \). By definition,\(^2\)

\[
\theta_a^k(q_a^k) = \begin{cases} 0 & \text{for } a \in \mathcal{P} \setminus \mathcal{P}, \\ \geq 0 & \text{for } a \in \mathcal{P}. \end{cases}
\]

Let \( \theta \) be a vector of link toll functions for all links and all time intervals. Similarly, the coefficient vector will be defined as \( w \). Let \( w \in \mathbb{W} \), where \( \mathbb{W} \) is a compact set, i.e.,

\[
\begin{align*}
& w_{a(m),k} \in \left[ w_{a(m),k}^{\min}, w_{a(m),k}^{\max} \right] \quad \text{for all } a \in \mathcal{P} \quad \text{and for all } m, \\
& w_{a(m),k}^{\min}, w_{a(m),k}^{\max} \in \mathbb{R}, \quad w_{a(m),k}^{\min} < w_{a(m),k}^{\max}. \quad \text{Note that while} \\
& \text{coefficients } w_{a(m),k} \text{ can be negative, the toll has to be} \\
& \text{nonnegative on all links, as stated in (3).}
\end{align*}
\]

With \( M = 0 \) in equation (2) the toll level becomes time-varying, but not directly dependent on traffic flow.

B. Drivers’ behavior – dynamic traffic assignment

The so-called dynamic traffic assignment (DTA) model describes user-optimal flows over a network in which each driver chooses his/her preferred route and his/her preferred departure time from origin to destination, based on the time-varying conditions in the network.

The standard DTA models consist of a dynamic travel choice (DTC) model and a dynamic network loading (DNL) model ([22], [20]).

The DTC contains a path choice model in which all travelers are distributed on all available routes so that some kind of dynamic user equilibrium is achieved.

In the problem of traffic assignment with given total traffic demand, each driver chooses both a departure time and a certain route from his/her origin to his/her destination in order to minimize his/her perceived travel costs. We assume that some Dynamic traffic equilibrium applies.\(^3\)

The dynamic network loading (DNL) model is formulated as a system of equations expressing link dynamics, flow conservation, flow propagation, and boundary constraints. The DNL model simulates the progression of the route flows on the network, yielding dynamic link flows, link volumes, and link travel times developing over time. The DNL model used in this paper is adapted from [22] and can be found in, e.g., [21].

C. The problem formulation

The goal of the road authority is to choose an optimal \( w^* \), minimizing his/her objective function. We will denote this function by \( Z = Z(q(w),w) \). The problem to be dealt with can be formulated as follows:

\[
(P) \quad \begin{cases}
\text{Find} & w^* = \arg \min_{w \in \mathcal{W}} Z(q(w),w), \\
\text{subject to} & q \in \text{DUE}(w),
\end{cases}
\]

where (2) and (3) hold.

We assume that the function \( Z \) is defined for each \( w, q(w) \) without any additional requirements. Therefore, a wide range of functions can be used as \( Z \).\(^4\)

The expression \( q \in \text{DUE}(w) \) reads as “link flow vector \( q \) is a result of a used dynamic user equilibrium (DUE) model when a polynomial toll function with coefficient vector \( w \) is used.” \(^\star\) Note that the function \( Z \) in the problem (P) is not specified, we only assume that it is a function of \( w \) of \( q(w) \).

The “standard” Stackelberg problem would be defined as a subproblem of (P), with \( M = 0 \).

\(^1\)Since some of the variables have to be rounded off, additional discussion about the consistency of these equations is needed. Such a discussion can be found in, e.g., [21].

\(^2\)More advanced toll functions can include, e.g., traffic flows from previous time periods, but we are looking for a very simple scheme improving the system performance. Therefore we will restrict ourselves to toll functions in the form (2). See [18] for discussions on this topic.

\(^3\)At this moment no further specification of such an equilibrium is required.

\(^4\)Typically, the function \( Z \) is defined as the total travel time of the system, negative of the total toll revenue, or negative of the surplus of the system.
D. General problem properties

Note that problem (P) is a nonlinear programming problem. Also, the problem (P) has at least one solution if the DUE constraint represents a compact set of \((w, q(w))\). If for any given \(w\) the set \(DUE(w)\) is a singleton, \(w \rightarrow q\) is a one-to-one mapping. In this case, the continuity of \(q\) with respect to \(w\) will guarantee that the constrained set of \((P)\) is closed, which implies the solution existence of \((P)\) since \(q\) and \(w\) are bounded. In general, \(DUE(w)\) may have multiple solutions in terms of \(q\) and thus \(DUE(w)\) may not be a singleton. In this case, \(DUE(w)\) is a point-to-set mapping of \(w\) \((\ref{eq:point_to_set})\). The solution existence of \((P)\) will depend on the compactness of the graph \(DUE(w)\), defined as

\[
Ψ(w, q) = \{ (w, q) | q \in DUE(w), \forall w \in W \}.
\]  \(\text{(4)}\)

Theorem 2.1: The problem \((P)\) has at least one solution if the following conditions are satisfied:

i. The set \(DUE(w)\) is nonempty and compact for \(\forall w \in W\).

ii. Let \(w, \overline{w} \in W\) and let \(q \in DUE(w)\), \(\overline{q} \in DUE(\overline{w})\). For each \(\epsilon > 0\), there exists \(\delta > 0\) such that if \(||w - \overline{w}|| < \delta\), then

\[
\max \min_{q \in DUE(w)} \min_{\overline{q} \in DUE(\overline{w})} ||q - \overline{q}|| < \epsilon.
\]

iii. The link travel cost functions on all links are continuous functions of the link flows on the same links.

Proof: Let \(R(0, \epsilon)\) be an open ball with radius \(\epsilon\). Then \(\mathcal{Y} \triangleq DUE(w) + R(0, \epsilon)\) is an open set containing \(DUE(w)\). Let us define an other open set \(\mathcal{Z} \triangleq \{ w : ||w - \overline{w}|| < \delta \}\) containing \(\overline{w}\). According to condition ii., in Theorem 2.1, for any \(\epsilon > 0\), there exists \(\delta > 0\) such that

\[
\max \min_{q \in DUE(w)} \min_{\overline{q} \in DUE(\overline{w})} ||q - \overline{q}|| < \epsilon,
\]

which is equivalent to \(\cup_{\overline{w} \in \mathcal{Z}} DUE(w) \subseteq \mathcal{Y}\). Thus, under ii., the point-to-set mapping of \(DUE(w)\) is upper-semicontinuous. Together with condition i., it implies that the point-to-set mapping \(DUE(w)\) is closed on set \(W\). Thus the graph \(Ψ(w, q)\) is closed. Also, under i., \(DUE(w)\) is bounded for any \(w \in W\). Since \(w\) is a bounded set, the graph \(Ψ(w, q)\) is bounded as well. Thus, graph \(w \in W\) is compact. Together with iii. and the fact that \(W\) is compact, we can conclude that (P) has at least one solution, since it is a nonlinear programming problem with a continuous objective function defined on a compact set.

Theorem 2.2: Problem \((P)\) is strongly NP-hard.

Proof: See \cite{18}.

Note that Theorems 2.1 and 2.2 extend theory about "standard" Stackelberg games (with \(M = 0\)), which can be found in, e.g., \cite{24}, \cite{25}. Additionally, the problem \((P)\) can have a nonunique solution (See \cite{18} for details).

III. PROBLEM SOLUTION

The solution method for \((P)\) that we propose is a combination of a neural networks approach for the upper level of the problem (finding optimal traffic-flow dependent toll for the road authority) and a method of sufficient averages (MSA) for the lower level of the problem (finding optimal traffic flows of the travelers according to the chosen dynamic user equilibrium). The concept of neural networks is closely related to the concept of supervised learning, which will be explained below. In Figure 1 the flow diagram of the solution process is depicted.

A. Supervised learning

Let function \(g : \mathbb{R}^n \rightarrow \mathbb{R}^m\) assign a vector \(\mathbf{y}^i \in \mathbb{R}^m\) to each vector \(\mathbf{x}^i \in \mathbb{R}^n\), i.e., \(\mathbf{y}^i = g(\mathbf{x}^i)\). We will refer to the pair \((\mathbf{x}^i, \mathbf{y}^i)\) as the \(i\)-th pattern of the function \(g\). The vector \(\mathbf{x}^i\) will be called the input vector (of \(g\)) and the vector \(\mathbf{y}^i\) will be called the output vector (of \(g\)). Supervised learning is a way to approximate the function \(g\) given a set of \(o\) patterns \(\mathcal{O}\).

An artificial neural network (ANN) can be thought of as a simple mathematical formula with parameters called weights \(\mathcal{O}\). The result of supervised learning is an approximation function \(g^{app}\) with an appropriately chosen vector of weights \(s\). The goal of supervised learning with ANN is to find a function \(g^{app} : \mathbb{R}^n \rightarrow \mathbb{R}^m\), that approximates the function \(g\) in the best way. Moreover, it is required that \(g^{app}\) has derivatives of all finite orders in the components of \(\mathbf{x}\).

Several criteria can be used to validate whether the function \(g^{app}\) is "close enough" to \(g\). In our approach the so-called validation error for each pattern \((\mathbf{x}^i, \mathbf{y}^i), i = 1, 2, \ldots, o,\) is
minimized.

The set of \( o \) patterns is divided into a set of \( t \) training patterns and a set of \( o - t \) validation patterns. For a given vector of weights \( s \) the training and the validation errors are calculated by

\[
e_t(s) = \frac{1}{2} \sum_{i=1}^{o} \sum_{k=1}^{m} (g_{k}^{app}(x^i,s) - y^i_k)^2, \\
e_v(s) = \frac{1}{2} \sum_{i=1}^{o} \sum_{k=1}^{m} (g_{k}^{app}(x^i,s) - y^i_k)^2,
\]

where \( g_{k}^{app} \) and \( y^i_k \), \( k = 1,2,\ldots,m \), refer to the \( k \)-th entry of \( g_{k}^{app} \) and \( y^i \), respectively. The elements of \( s \) are optimized only for \( t \) training patterns, while the validation patterns are used to prevent overtraining.

An ANN is trained iteratively, i.e., \( e_t \) is decreased by adaption of \( s \), until \( e_t \) increases for two consecutive iterations (prevention of overtraining). Note that the training stops before a local minimum of \( e_t \) is reached. Weight upgrades \( s^{iter+1} \rightarrow s^{iter} \) can be calculated with any minimization algorithm, e.g., a first derivative method such as steepest descent, or a second derivative method such as Newton’s method. For the first derivative methods the iterative sequence

\[
s^{iter+1} = s^{iter} + \eta \left( e_t(s^{iter}), \nabla s e_t(s^{iter}) \right) \Delta s \left( e_t(s^{iter}), \nabla s e_t(s^{iter}) \right),
\]

with the search direction \( \Delta s \) and with step length \( \eta \) takes place. For the neurosimulation the FAUN \(^5\) 1.0 simulator is used ([26]). Numerical methods implemented within FAUN 1.0 for constrained nonlinear least-squares problems ([27]) are sequential quadratic programming (SQP) methods and generalized Gauss-Newton (GGN) methods. These methods can exploit the special structure of the Hessian matrix of \( e_t \) ([28], [29], [30]). The SQP and GGN methods can automatically overcome most of the training problems of ANN such as flat spots or steep canyons of the error function \( e_t \). See [26], [31], [32] for more information about neurosimulator FAUN 1.0.

B. Solution of problem (P)

The solution process consists of the following steps:

1) For every selection of the toll vector \( w \) the so-called C-load algorithm ([22]) is used in order to obtain the DUE traffic flows \( q \in DUE(w) \). The value of the objective function \( Z(q(w),w) \) is computed.

2) The set of sample points \( (w, Z(q(w),w)) \) is divided into training patterns and validation patterns and the neurosimulation introduced in Section III-A takes place in order to approximate \( Z \).

3) This approximation function of \( Z \) is minimized, optimal \( w \) minimizing \( Z \) is found.

For description of the solution process in more detail, see [18].

IV. CASE STUDIES

In this section the performance of the algorithm introduced in Section III will be tested on more specific problem of the type (P). We will consider the so-called Chen network, depicted in Fig. 2. In the case studies we will confine ourselves by the following assumptions:

- The objective function for the road authority is the total travel time, i.e., \( Z \) is the total travel time, \( f_t \).
- We consider two origin-destination pairs: (1,5), (3,5).
- The input parameters are set as follows: \( \mathcal{X} = \{1,\ldots,16\} \), \( \alpha_t = 10 \) [euro/h], \( \alpha_o = 0.8 \) [euro/h], \( \alpha_o = 2 \) [euro/h], \( \hat{k}_{i(1,5)} \) dep = 4, \( \hat{k}_{i(3,5)} \) dep = 6, \( \hat{k}_{i(3,5)} \) arr = 8, \( d^{(1,5)} = 8000 \) [veh], \( d^{(3,5)} = 4000 \) [veh], \( \epsilon_o = 0.001 \), \( w^{(m)}_{\min} = -10 \), \( w^{(m)}_{\max} = 10 \).
- We consider the so-called dynamic logit-based user equilibrium as a specific dynamic user equilibrium (see [18] for more details), with \( \mu = 0.2 \). Convergence of the algorithm is verified using the relative dynamic duality gap (see [18] for its definition). If the relative duality gaps of two consecutive iterations are close enough, i.e., \( |e^{(iter)} - e^{(iter-1)}| < \epsilon_{\max} \), with a given small positive number \( \epsilon_{\max} \), the algorithm is terminated. We set \( \epsilon_{\max} \) to \( 10^{-3} \).
- For each directed arc \( a \in \mathcal{X} \) the following parameters are initially given: link length \( l_{aa} \) [km], maximum speed \( v_{\text{max}}^{\text{aa}} \) [km/h], minimum speed \( v_{\text{min}}^{\text{aa}} \) [km/h], critical speed \( v_{\text{crit}}^{\text{aa}} \) [km/h], jam density \( j_{\text{jam}}^{\text{aa}} \) [pcu/km], where pcu denotes passenger car units, and the unrestricted link capacity \( C_a \) [pcu/h]. Dynamic link travel time for an individual user entering link \( a \) during the \( k \)-th time interval \( (k \in \mathcal{X}) \) is defined as \( t_a^{(k)} = \frac{1}{v_a} \), where the link speed \( v_a^{(k)} \) is defined as Smudlers speed-density function (see [33]):

\[
v_a^{(k)} = \begin{cases} 
\frac{\phi_{max}^{(k)} + \phi_{max}^{(k-1)} - \phi_{min}^{(k)} - \phi_{min}^{(k-1)}}{\phi_{max}^{(k)} - \phi_{min}^{(k)}} j_{\text{arr}}^{(k)}, & \text{if } j_a^{(k)} \leq j_{\text{crit}}^{(k)} \\
\phi_{\text{min}}^{(k)} + (\phi_{\text{max}}^{(k)} - \phi_{\text{min}}^{(k)}) \left( \frac{\phi_{\text{arr}}^{(k)}}{\phi_{\text{min}}^{(k)}} \right), & \text{if } j_a^{(k)} \geq j_{\text{min}}^{(k)}
\end{cases}
\]

with critical density \( j_{\text{crit}}^{(k)} \) [pcu/km] defined as \( j_{\text{crit}}^{(k)} = C_a / v_{\text{crit}}^{(k)} \).

- Initial link parameters are depicted in Table I.
- Link tolls are defined by (2). The results of the problems with \( M = 0,1,2,3 \) were compared. The algorithm introduced in Section III was applied to solve the problem.

\(^5\)Fast Approximation with Universal Neural networks
### TABLE I
**INITIAL LINK PARAMETERS**

<table>
<thead>
<tr>
<th>α</th>
<th>s_w</th>
<th>( t_{\text{max}} )</th>
<th>( t_{\text{max}} )</th>
<th>( t_{\text{min}} )</th>
<th>( t_{\text{min}} )</th>
<th>( C_{\text{p}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.5</td>
<td>150</td>
<td>90</td>
<td>20</td>
<td>50</td>
<td>3500</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>120</td>
<td>70</td>
<td>10</td>
<td>150</td>
<td>1500</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>120</td>
<td>70</td>
<td>10</td>
<td>150</td>
<td>1500</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>150</td>
<td>90</td>
<td>20</td>
<td>50</td>
<td>3500</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>120</td>
<td>70</td>
<td>10</td>
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<td>1500</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
<td>120</td>
<td>70</td>
<td>10</td>
<td>150</td>
<td>1500</td>
</tr>
</tbody>
</table>

### TABLE II
**THE RESULTING TOLL VALUES (FOR \( k \in \{1, \ldots, 8\} \)) AND TOTAL TRAVEL TIMES FOR CASE STUDY WITH ONE LINK TOLLED**

<table>
<thead>
<tr>
<th>M</th>
<th>( \theta^{(1)} ) [( \text{euro} )]</th>
<th>Z [h]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.4</td>
<td>4.2</td>
</tr>
<tr>
<td>1</td>
<td>1.9</td>
<td>2.3</td>
</tr>
<tr>
<td>2</td>
<td>1.8</td>
<td>2.3</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>1.9</td>
</tr>
</tbody>
</table>

Fig. 3. The resulting traffic flows for case study with two links tolled, \( M = 3 \).

### A. Case study with one link tolled

We will first consider the situation with only link 1 tolled, \( M \) will vary from 0 to 3. For the solution process introduced in Section III 40000 samples \((w, Z(q(w), w))\) were collected. For each computations, these samples were splitted into 30000 training data and 10000 validation data.

The resulting toll values for first 8 time intervals and values of total travel times for the situation with \( M = 0, M = 1, M = 2, \) and \( M = 3 \) are depicted in Table II. Clearly, the polynomial tolls result in a better outcome for the road authority. For all variants of \( M \) the approximation function had many local minima, but only one global minima.

The computation time on 16 processors was approximately 10.5 [min]. This time can be geometrically decreased by further parallelization.

### B. Case study with two links tolled

Let both links 1 and 4 be tolled. We will use 80000 samples, with 60000 training data and 20000 validation data. In Table III the resulting toll values for first 8 time intervals and resulting total travel times for the situations with \( M = 0, M = 1, M = 2, \) and \( M = 3 \) are depicted. Again, the polynomial tolls improved the system performance remarkably.

Let \( p_1 \) denote the route consisting of links 1 and 4, let \( p_2 \) denote the route consisting of links 1, 5, and 6, let \( p_1 \) denote the route consisting of links 2, 3, and 4, and let \( p_4 \) denote the route consisting of links 2, 3, 5, and 6. In Figure 3 the traffic flows on the routes \( p_1, p_2, p_3, \) and \( p_4 \) are depicted. Note that quite many travelers use the route \( p_1 \), as its costs are lower even after imposing tolls on both its links.

The computation time on 16 processors was approximately 52.3 [min].

### C. Discussion

After application of the neurosimulation-based algorithm to solve the problems presented in this section we applied a very detailed grid search on the same problems. It showed that the obtained minimal values differed by 0.5 percent maximally (by means of \( L_2\)-norm) and the global minima were located in the same points as those computed using the neurosimulation-based algorithm. One has to be very careful with choosing the right value of the validation error as well as the sample area, though, to obtain correct results.

The tolls set as polynomial functions of traffic flows improved the system performance remarkably. Note that while these tolls are computed as functions of traffic flows, they are imposed as positive values.

### V. CONCLUSIONS & FUTURE RESEARCH

We formulated a very general version of the dynamic optimal toll design problem and discussed its properties. As the problem is NP-hard and the objective function of the road authority generally has many local minima, we proposed a neurosimulation-based approximation algorithm using Neurosimulator FAUN 1.0 to deal with the problem.

The algorithm that we proposed is applicable to a wide range of optimal toll design problems, with different DUE formulations, and with a wide range of objective functions.
The performance of the proposed algorithm was tested on a case study on the Chen network.

As the proposed solution method is an approximation method, it is important to choose the initial parameters for neurosimulation very carefully to ensure sufficient accuracy, as well as to avoid overtraining. These issues are discussed in [26], [18]. With a proper choice of the criteria for neurosimulation the algorithm performs very well, as we showed in the case studies.

Even solving a problem defined on a small network was quite time consuming. However, the algorithm can be parallelized, and, therefore, this pitfall can be easily eliminated and the algorithm can be applied on real-size networks.

We showed that the traffic-flow dependent tolls can perform remarkably better than the traffic-flow invariant tolls, even with simple polynomial toll functions. Use or more sophisticated toll functions may bring even better results.

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