Stackelberg versus inverse Stackelberg games in the dynamic optimal toll design problem

Which approach provides more efficient tolling?

TRAIL Research School, Delft, November 2006

Authors

Ing. Kateřina Staňková, Prof. dr. Geert Jan Olsder

Faculty of Electrical Engineering, Mathematics and Computer Science, Mathematical Systems Theory, Delft University of Technology

Dr. Michiel Bliemer

Faculty of Civil Engineering and Geosciences, Transport and Planning, Delft University of Technology

Contents

Abstract

In this paper, the dynamic optimal toll design problem is considered as a one-leader-more-followers hierarchical non-cooperative game. On a given network the road authority as the leader tolls some links in the network to minimize the total travel time of the system, while travelers as followers are assumed to be driven by the dynamic route choice equilibrium assignment. So far toll has always been considered as constant or time-varying. Inspired by San Diego's Interstate 15 congestion pricing project, in which heuristics with toll proportional to traffic flow are applied on a real two-link highway network, we construe toll as proportional to traffic flows in the network. If toll is set either as constant or time-varying, the dynamic optimal toll design problem can be treated as a Stackelberg game; if toll is set as a function of the traffic flows in the network, the problem can be treated as an inverse Stackelberg game.

On a simple two-link network we investigate various tolling concepts, representing both Stackelberg and inverse Stackelberg games.

We show that the use of flow-dependent toll can improve system performance better than the use of either constant or time-varying toll, provided that the tolling scheme is chosen properly.

Keywords

Road pricing, dynamic optimal toll design problem, Stackelberg games, inverse Stackelberg games

1 Introduction

Road pricing is one of the most efficient methods to avoid congestion problems on road networks (Verhoef (2002a), A. D. May (2000)). With the use of appropriate tolls the road authority can force travelers to behave so as to improve the performance of a given traffic system. This led to the introduction of the so-called Optimal Toll Design Problem.

Some researchers have attempted to solve the problem by means of a game-theoretic approach, in which the game is of a Stackelberg type (e.g. Verhoef (2002b), Joksimovič et al. (2004), Chan & Lam (2005)). However, these approaches consider toll as independent of link and route traffic flow, while it seems to be reasonable to introduce toll as proportional to traffic flow.

This is why we construed (Staňková et al. (2006b)) the static optimal toll design problem as an inverse Stackelberg game (Olsder (2005)), where the link tolls are functions of link or route traffic flows in the network. We considered travelers driven by the deterministic user equilibrium and solved the problem analytically. We proved that with the static optimal toll design problem defined on a two-link network there exists an inverse Stackelberg strategy ensuring a better outcome for the road authority than with the standard Stackelberg strategy. In Staňková et al. (2006a) we applied an inverse Stackelberg game to a dynamic optimal toll design problem with very simple tolling schemes.

This approach is inspired by San Diego's Interstate 15 congestion pricing project (Supernak et al. (2002)), in which the two-link highway network was studied. One of the links was tolled according to network occupancy and toll was established in a heuristic manner. We consider case studies with a two-link network as well, the problem now being dynamic. The aim of the road authority is to minimize the total travel time of the network by tolling one of the links, whereas travelers choose a link so as to minimize their travel costs. As a reference case of the dynamic optimal toll design problem we take Stackelberg games with toll being constant and time-varying, respectively. We analytically compute the results of inverse Stackelberg games differing in the tolling schemes used and compare them with the results of the reference cases. We discuss whether the inverse Stackelberg approach defining toll as being a function of traffic flows leads to a better outcome for the road authority than the Stackelberg approach with toll being either constant or time-varying.

Finding an optimal toll as a function of traffic flows in the network will help solve the congestion problems by road pricing more efficiently.

The content of the paper is as follows: In Section 2 we briefly introduce Stackelberg and inverse Stackelberg game schemes with a clear emphasis on the not-so-well-known inverse Stackelberg games. In Section 3 the dynamic bilevel optimal design problem is defined as a Stackelberg and inverse Stackelberg game, respectively. In Section 4 we analytically solve the case studies on a small network. In the considered function spaces the best possible link toll functions for the road authority are found and the results of these games compared with the ones of a traditional Stackelberg game with constant or time-varying toll. The obtained results and possibilities of future research are discussed in Section 5.

2 Stackelberg and Inverse Stackelberg Equilibria and Terminology

In this section we introduce the basic concept of Stackelberg game (SG) and inverse Stackelberg game (ISG), in Section 2.1 for one-leader-one-follower games and in Section 2.2 for one-leader-more-followers games.

2.1 One Leader - One Follower

In its simplest form, there are two players, called leader ($\mathcal{L}$) and follower ($\mathcal{F}$), respectively, either having his own cost function,

$$\mathcal{J}_{\mathcal{L}}(u_{\mathcal{L}}, u_{\mathcal{F}}),\ \mathcal{J}_{\mathcal{F}}(u_{\mathcal{L}}, u_{\mathcal{F}}),$$

where $u_{\mathcal{L}}, u_{\mathcal{L}} \in \mathrm{R}$ are $\mathcal{L}$'s and $\mathcal{F}$'s decision variables, respectively. Either player wants to choose his own decision variable in such a way as to minimize his own cost function. Without stating an equilibrium concept, the problem as stated so far is not well defined. Such an equilibrium concept could, for instance, be one named after Nash or Pareto (Başar & Olsder (1999)).

2.1.1 Stackelberg Game

According to the Stackelberg equilibrium concept $\mathcal{L}$ announces his decision $u_{\mathcal{L}}$, which is subsequently made known to $\mathcal{F}$. With this knowledge, $\mathcal{F}$ chooses his $u_{\mathcal{F}}$. Hence $u_{\mathcal{F}}$ becomes a function of $u_{\mathcal{L}}$, written as $u_{\mathcal{F}} = l_{\mathcal{F}}(u_{\mathcal{L}})$, determined by the relation

$$\min_{u_{\mathcal{F}}} \mathcal{J}_{\mathcal{F}}(u_{\mathcal{L}}, u_{\mathcal{F}}) = \mathcal{J}_{\mathcal{F}}(u_{\mathcal{L}}, l_{\mathcal{F}}(u_{\mathcal{L}})). \tag{1}$$

It is assumed that this minimum exists and that it is unique for each possible choice $u_{\mathcal{L}}$ by the leader. Before the leader announces his decision $u_{\mathcal{L}}$, he will realize how the follower will react and hence the leader will choose, and subsequently announce, $u_{\mathcal{L}}$ so as to minimize $\mathcal{J}_{\mathcal{L}}(u_{\mathcal{L}}, l_{\mathrm{F}}(u_{\mathcal{L}}))$.

Example 1

Suppose $\mathcal{J}_{\mathcal{L}}(u_{\mathcal{L}}, u_{\mathcal{F}}) = (u_{\mathcal{F}} - 5)^2 + u_{\mathcal{L}}^2,\ \mathcal{J}_{\mathcal{F}}(u_{\mathcal{L}}, u_{\mathcal{F}}) = u_{\mathcal{L}}^2 + u_{\mathcal{F}}^2 - u_{\mathcal{L}}u_{\mathcal{F}}$. *The reaction curve* $l_{\mathcal{F}}$ *is given by* $u_{\mathcal{F}} = \frac{1}{2}u_{\mathcal{L}}$ *and hence* $u_{\mathcal{L}}$ *will be chosen so as to minimize*

$$\left(\frac{1}{2}u_{\mathcal{L}} - 5\right)^2 + u_{\mathcal{L}}^2,$$

which immediately results in $u_{\mathcal{L}} = 2$. *With this decision by the leader the follower will choose* $u_{\mathcal{F}} = 1$. *The costs for the players are given by 20 and 3, respectively.*

2.1.2 Inverse Stackelberg Game

Another equilibrium concept, to be dealt with now, is the *inverse Stackelberg equilibrium* as introduced in (Olsder (2005)). The leader does not announce the scalar $u_{\mathcal{L}}$, as above, but a function $\gamma_{\mathcal{L}}(\cdot)$ which maps $u_{\mathcal{F}}$ into $u_{\mathcal{L}}$. Given the function $\gamma_{\mathcal{L}}(\cdot)$, the follower will make his choice $u_{\mathcal{F}}$ according to

$$u_{\mathcal{F}}^{*} = \arg\min_{u_{\mathcal{F}}} \mathcal{J}_{\mathcal{F}}(\gamma_L(u_{\mathcal{F}}), u_{\mathcal{F}}).$$

Optimal quantities will be flagged by an asterisk. The leader, before announcing his $\gamma_{\mathcal{L}}(\cdot)$, will of course realize how the follower will play and he should exploit this knowledge in order to choose the best possible γ-function, such that, ultimately, his own cost function $\mathcal{J}_{\mathcal{L}}$ becomes as small as possible. Symbolically we could write

$$\gamma_{\mathcal{L}}^{*}(\cdot) = \arg\min_{\gamma_{\mathcal{L}}(\cdot)} \mathcal{J}_{\mathcal{L}}(\gamma_L(u_{\mathcal{F}}^{*}(\gamma_{\mathcal{L}}(\cdot))), u_{\mathcal{F}}^{*}(\gamma_{\mathcal{L}}(\cdot))).$$

In this way one enters the field of composed functions, which is known to be a notoriously complex area. From here onward it turns out to be diffult to proceed in an analytic way. However, the following example shows a trick that often works.

Example 2

Suppose the cost functions are those of Example 1. If both the follower and the leader would be so kind as to minimize $\mathcal{J}_{\mathcal{L}}(u_{\mathcal{L}}, u_{\mathcal{F}})$, the follower totally disregarding his own cost function, the leader would obtain his so-called team minimum, *i.e.,* $\mathcal{J}_{\mathcal{L}}(0, 5) = 0$. *Now the leader should choose the curve $u_{\mathcal{L}} = \gamma_{\mathcal{L}}(u_{\mathcal{F}})$ in such a way that the minimum $u_{\mathcal{L}} = 0,\ u_{\mathcal{F}} = 5$ lies on this curve and such that this curve does not have other points in common with the set*

$$\mathcal{J}_{\mathcal{F}}(u_{\mathcal{L}}, u_{\mathcal{F}}) = u_{\mathcal{L}}^2 + u_{\mathcal{F}}^2 - u_{\mathcal{L}}u_{\mathcal{F}} \leq \mathcal{J}_{\mathcal{F}}(0, 5) = 25.$$

An example of such a curve is $u_{\mathcal{L}} = 2u_{\mathcal{F}} - 10$. With this choice by the leader, the best for the follower to do is to minimize

$$\mathcal{J}_{\mathcal{F}}(2u_{\mathcal{F}} - 10, u_{\mathcal{F}}),$$

which leads to $u_{\mathcal{F}} = 5$. Hence $u_{\mathcal{L}} = 0$ and, to our surprise, the leader has obtained his team minimum in spite of the fact that the follower minimized his own cost function (however, subject to the constraint $u_{\mathcal{L}} = \gamma_{\mathcal{L}}(u_{\mathcal{F}}) = 2u_{\mathcal{F}} - 10$).

Other examples exist in which the leader cannot obtain his team minimum, and such problems are harder to deal with as exemplified by the following example.

Example 3

This is a continuation of Example 2. We add the constraints $-4 \leq u_{\mathcal{L}} \leq +3$ and $-5 \leq u_{\mathcal{F}} \leq 7$ which the two players must obey. The worst that can happen to the follower is characterized by $\min_{u_{\mathcal{F}}} \max_{u_{\mathcal{L}}} \mathcal{J}_{\mathcal{F}}$ which is realized for $u_{\mathcal{F}} = -2$, $u_{\mathcal{L}} = -4$ resulting in $\mathcal{J}_{\mathcal{F}}(-4, -2) = 12$. Consider the team problem

$$\min_{u_{\mathcal{L}}, u_{\mathcal{F}}} \mathcal{J}_{\mathcal{L}}, \text{ subject to } \mathcal{J}_{\mathcal{F}} \leq \mathcal{J}_{\mathcal{F}}(-4, -2) = 12.$$

The calculation to find this solution, though straightforward, does not lead to a "nice" answer. The solution will simply be indicated by $u_{\mathcal{L}}^{\dagger}, u_{\mathcal{F}}^{\dagger}$. An optimal choice for the leader is

$$u_{\mathcal{L}} = \gamma_{\mathcal{F}} = \begin{cases} -4 & \text{for } -5 \leq u_{\mathcal{F}} < u_{\mathcal{F}}^{\dagger} - \epsilon, \\ u_{\mathcal{L}}^{\dagger} & \text{for } u_{\mathcal{F}}^{\dagger} - \epsilon \leq u_{\mathcal{F}} \leq 7, \end{cases}$$

where ϵ is an arbitrarily small positive number. The quantity ϵ being greater than zero makes the solution unique; $\epsilon = 0$ would lead to a non-unique response by the follower.

2.2 More Followers

In Section 2.1 we introduced the basic concept of ISG and SG. Since in the dynamic optimal toll design problem we deal with one-leader-more-followers cases of the games, this section deals with an extension of Section 2.1 to a situation with more followers. Let us assume now a noncooperative game with one *leader* ($\mathcal{L}$) and $m \in \mathbb{N}$ *followers* ($\mathcal{F}_1, \ldots, \mathcal{F}_m$) ($m \in \mathbb{N}$, $m \geq 2$).[1] The leader has decision variables $u^1_{\mathcal{L}}, \ldots, u^n_{\mathcal{L}} \in \mathbb{R}^n$ ($n \in \mathbb{N}$) and the i-th follower $\mathcal{F}_i$ ($i \in \{1, \ldots, m\}$) has the decision variable $u_{\mathcal{F}_i} \in \mathbb{R}$. We will denote vector $(u^1_{\mathcal{L}}, \ldots, u^n_{\mathcal{L}}) \in \mathbb{R}^n$ by $\overline{u}_{\mathcal{L}}$. The leader has the cost function $\mathcal{J}_{\mathcal{L}}(\overline{u}_{\mathcal{L}}, \overline{u}_{\mathcal{F}})$ with $\overline{u}_{\mathcal{F}} = (u_{\mathcal{F}_1}, \ldots, (u_{\mathcal{F}_i}, \ldots u_{\mathcal{F}_m})$, each $\mathcal{F}_i$ has the cost function $\mathcal{J}_{\mathcal{F}_i}(\overline{u}_{\mathcal{L}}, \overline{u}_{\mathcal{F}})$. Each player wants to minimize his own cost function.

2.2.1 Stackelberg Game

According to the Stackelberg equilibrium concept $\mathcal{L}$ announces his decisions $\overline{u}_{\mathcal{L}} \in \mathbb{R}^n$, which are subsequently made known to the followers. With this knowledge, each follower $\mathcal{F}_i$ chooses his $u_{\mathcal{F}_i} \in \mathbb{R}$. Hence each $u_{\mathcal{F}_i}$ becomes a function of $\overline{u}_{\mathcal{L}}$, written as $u_{\mathcal{F}_i} = l_{\mathcal{F}_i}(\overline{u}_{\mathcal{L}})$, which is determined by the relation

$$\min_{u_{\mathcal{F}_i}} \mathcal{J}_{\mathcal{F}_i}(\overline{u}_{\mathcal{L}}, \overline{u}_{\mathcal{F}}) = \mathcal{J}_{\overline{u}_{\mathcal{F}_i}}(\overline{u}_{\mathcal{L}}, u_{\mathcal{F}_1}, \ldots, u_{\mathcal{F}_{i-1}}, l_{\mathcal{F}_i}(\overline{u}_{\mathcal{L}}), u_{\mathcal{F}_{i+1}}, \ldots, u_{\mathcal{F}_m}). \qquad (2)$$

It is assumed that this minimum exists and that it is unique for each possible choice $\overline{u}_{\mathcal{L}}$ by the leader. Before the leader announces his decisions $\overline{u}_{\mathcal{L}}$, he will realize how the follower will react and hence the leader will choose, and subsequently announce, $\overline{u}_{\mathcal{L}}$ so as to minimize $\mathcal{J}_{\mathcal{L}}\left(\overline{u}_{\mathcal{L}}, l_{\mathcal{F}_1}\left(\overline{u}_{\mathcal{L}}\right), \ldots l_{\mathcal{F}_m}\left(\overline{u}_{\mathcal{L}}\right)\right)$.

2.2.2 Inverse Stackelberg Game

With the use of the *inverse Stackelberg equilibrium* concept between the leader and the followers $\mathcal{L}$ announces his decision variables $\overline{u}_{\mathcal{L}}$ as the vector of the functions $\overline{\gamma(\cdot)} = (\gamma_1(\cdot), \gamma_2(\cdot), \ldots \gamma_n(\cdot))$, respectively, where each $\gamma_i(\cdot)$ is a continuous mapping from $\overline{u}_{\mathcal{F}} = (u_{\mathcal{F}_1}, \ldots, u_{\mathcal{F}_m})$ into $u^i_{\mathcal{L}}$. Given the vector $\overline{\gamma(\cdot)}$, $\mathcal{F}_i$ will make his choice $u_{\mathcal{F}_i}$ according to $u^*_{\mathcal{F}_i} = \arg\min_{u_{\mathcal{F}_i}} \mathcal{J}_{\mathcal{F}_i}(\overline{\gamma(\cdot)}, \overline{u}_{\mathcal{F}})$. $\mathcal{L}$, before announcing $\overline{\gamma(\cdot)}$, will of course realize how the followers will play and he should exploit this knowledge in order to choose optimal γ_i-functions, such that ultimately his own cost function $\mathcal{J}_{\mathcal{L}}$ becomes as low as possible. Symbolically we could write

$$\gamma^*_i(\cdot) = \arg\min_{\gamma_i(\cdot)} \mathcal{J}_{\mathcal{L}}\left(\gamma_1(\overline{u}^*_{\mathcal{F}}), \ldots, \gamma_n(\overline{u}^*_{\mathcal{F}}), \overline{u}^*_{\mathcal{F}}\right)$$

with $\overline{u}^*_{\mathcal{F}} = (u^*_{\mathcal{F}_1}, \ldots, u_{\mathcal{F}^*_m})$.

Remark 2.1 *The philosophy in most cases (as in the current paper) is first to get an impression of what the leader can achieve and subsequently try to find a strategy to really reach this goal. If one does not have any clue as to what the leader can obtain (in terms of minimal costs), hardly anything is known.*

[1]We assume that the followers $\mathcal{F}_1, \ldots, \mathcal{F}_m$ are among themselves driven by an equilibrium concept as well, for instance Nash, Pareto (Başar & Olsder (1999)), or Wardrop (Patriksson (1999)).

3 Dynamic Optimal Toll Design Problem

3.1 Problem Definition

Let $\mathcal{K} = \{1, 2, \ldots, |\mathcal{K}|\}$ ($|\mathcal{K}| \in \mathbb{N}$) be a time index set. The k-th and k'-th time intervals will be identified by $k \in \mathcal{K}$ and $k' \in \mathcal{K}$, respectively. Let $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ be a given strongly connected road network with a finite at least two-element node set $\mathcal{N}$ and a set $\mathcal{A} = \{l_1, \ldots, l_{|\mathcal{A}|}\}$ ($|\mathcal{A}| \in \mathbb{N}$) of directed links (arcs). Let $\mathcal{T} \subset \mathcal{A}$ be a set of tollable links. Let $\mathcal{OD} \subset \mathcal{N} \times \mathcal{N}$ be a set of origin-destination pairs. We will denote the nonempty set of simple routes from an origin o to a destination d by $\mathcal{P}^{(o,d)}$, and the set of all simple routes in the network by $\mathcal{P}$. Let $D^{(o,d),k}$ be the average departure rate of travelers departing during time interval k from origin o to destination d. For the sake of simplicity $D^{(o,d),k}$ is assumed to be inelastic and given. The average link flow rate of travelers entering link $l_j \in \mathcal{A}$ during time interval k will be denoted by $q^k_{l_j}$, the average route flow rate of travelers departing during time interval k along route $r_i \in \mathcal{P}$ will be denoted by $f^k_{r_i}$. For the sake of simplicity we will through this paper often talk about the link flow and the route flow instead of the average link flow rate and the average route flow rate, respectively. The link travel time on link $l_j \in \mathcal{A}$ for travelers entering link l_j during time interval k will be denoted by $\tau^k_{l_j}$ and defined as

$$\tau^k_{l_j} = \beta_{l_j} x^k_{l_j} + \delta_{l_j}. \tag{3}$$

Here β_{l_j} and δ_{l_j} are positive constants and $x^k_{l_j}$ is the number of travelers on link l_j (the link volume) at the beginning of time interval k, defined as

$$x^k_{l_j} = \sum_{\tau=1}^{k} q^\tau_{l_j} - \sum_{\zeta \in \mathcal{W}^k_j} q^\zeta_{l_j}, \tag{4}$$

where $\mathcal{W}^k_j = \{w | w + \tau^w_{l_j} \leq k\}$. We assume $x^1_{l_j} = q^1_{l_j}$.

To ensure the feasibility of the route flows $[f^k_{r_i}]_{r_i \in \mathcal{P}^{(o,d)}, k \in \mathcal{K}}$ with respect to the average departure rate of travelers $D^{(o,d),k}$ and the nonnegativity of the route flows, the following conditions have to be satisfied:

$$\sum_{r_i \in \mathcal{P}^{(o,d)}} f^k_{r_i} = D^{(o,d),k}, \quad (o,d) \in \mathcal{OD}, \quad k \in \mathcal{K}, \tag{5}$$

$$f^k_{r_i} \geq 0, \quad r_i \in \mathcal{P}^{(o,d)}, \quad (o,d) \in \mathcal{OD}, \quad k \in \mathcal{K}. \tag{6}$$

Let $[\delta^{k,k'}_{r_i,l_j}]_{r_i \in \mathcal{P}, l_j \in \mathcal{A}, k,k' \in \mathcal{K}}$ be a dynamic link-route incidence identifier for G with

$$\delta^{k,k'}_{r_i,l_j} = \begin{cases} 1, \quad \text{if travelers entering the route} \quad r_i \in \mathcal{P}^{(o,d)} \\ \text{during time interval} \quad k \quad \text{enter the link} \quad l_j \in \mathcal{A} \quad \text{during time interval} \quad k', \\ 0 \quad \text{otherwise.} \end{cases}$$

The average link flow rate $q^{k'}_{l_j}$ ($k' \in \mathcal{K}$) is defined by the average route flow rates through

$$q^{k'}_{l_j} = \sum_{k \in \mathcal{K}} \sum_{r_i \in \mathcal{P}} \delta^{k,k'}_{r_i,l_j} f^k_{r_i}, \quad l_j \in \mathcal{A}.$$

With each link $l_j \in \mathcal{A}$ we associate the link travel cost $c^k_{l_j}$ for travelers entering l_j during time interval k defined as $c^k_{l_j} = \alpha\tau^k_{l_j} + \theta^k_{l_j}$, where $\tau^k_{l_j}$ is the link travel time on the link l_j, α is the travelers' value of time (VOT), and $\theta^k_{l_j}$ is the link toll paid by travelers entering the link l_j during the time interval k. Route costs are additive. Let $\overline{\theta}^k$ be the $|\mathcal{T}|-$vector of nonnegative tolls on all tollable links during time interval $k \in \mathcal{K}$.

The travelers are driven by the *dynamic route choice equilibrium assignment model*, which is based on the assumption that all road users have complete and accurate information about the current traffic conditions, and that they choose among the shortest routes available. In an equilibrium state, for each origin-destination pair and for each departure time interval, the actual route costs on all used routes are equal (Bliemer (2001)).

3.2 The Dynamic Optimal Toll Design Problem from a Game-theoretic Viewpoint

The bilevel optimal dynamic design problem can be defined as the one-leader-more-followers SG and ISG, resp., as follows:

- The leader ($\mathcal{L}$) is the road authority minimizing the total travel time of the system by tolling tollable links, which can be symbolically written as

$$\overline{\theta^*}^k = \arg\min_{\overline{\theta}^k} \sum_{k=1}^{\mathcal{K}} \sum_{r_i \in \mathcal{P}} \tau^k_{r_i} f^k_{r_i}, \quad \forall k \in \mathcal{K}. \tag{7}$$

- The followers $\mathcal{F}_1, \ldots \mathcal{F}_m$ are the travelers on the road network. The decision variables of the travelers are the travelers' route choices, i.e., $u_{\mathcal{F}_i} \in \mathcal{P}^{(o,d)}$ if $\mathcal{F}_i$ travels from origin o to destination d. The travelers' decisions are driven by the dynamic route choice equilibrium assignment and they result in the link and route flows in the network for each time period, determining the route and the link volumes in the network.

If the dynamic optimal toll design problem is solved as a SG, then each $\theta^k_{l_j}$ ($l_j \in \mathcal{T}$, $k \in \mathcal{K}$) is set as a nonnegative number. In the case of uniform tolling $\theta^k_{l_j} = \theta_{l_j} \in \mathbb{R}^0_+$ for each $k \in \mathcal{K}$, $l_j \in \mathcal{T}$.

If the dynamic optimal toll design problem is solved as an ISG, tolls on tollable links are set as functions of link flow rates in the network, i.e., $\theta^k_{l_j} = \gamma_j(q^1_{l_1}, q^1_{l_2}, \ldots q^k_{l_{|\mathcal{A}|}})$, $l_j \in \mathcal{T}$. The goal of the road authority is to impose the best possible γ_j-functions ensuring the lowest possible total travel time,i.e., $(\overline{\theta}^k)^* = \overline{\gamma_j}^*(\cdot)$, where $\overline{\gamma_j}(\cdot)$ is the vector of γ_j-functions on all tollable links.

Remark 3.1 *In general, it can be quite difficult to find optimal γ_j-functions being functions of all the link flows in the network. This is why we define simple tolling functions, for example as $\theta^k_{l_j} = \xi_j\, q^k_{l_j}, \quad l_j \in \mathcal{T}$ or $\theta^k_{l_j} = \mu_j\, x^k_{l_j}, \quad l_j \in \mathcal{T}$, where $\xi_j, \mu_j \in \mathbb{R}_+$.*

Will the road authority be better off playing ISG than playing a SG strategy? In Staňková et al. (2006a) we proved that for a two-link network in the static case the answer is yes. In the following section we will discuss this question for the dynamic case.

4 Case studies

Let us assume a network with one origin-destination pair (o, d) and two links l_1, l_2 from o to d, where l_1 is tolled and l_2 is untolled. We consider 7 time intervals, i.e., $\mathcal{K} = \{1, 2, 3, 4, 5, 6, 7\}$. The average departure rates for each time interval are defined as depicted in Table 4, $\alpha = 10$ [euro/time interval], $\beta_{l_1} = \frac{1}{2000}$, $\beta_{l_2} = \frac{1}{1000}$, $\delta_{l_1} = 1$, $\delta_{l_2} = 2$.

Table 1: The average departure rates

$D^{(o,d),1}$	$D^{(o,d),2}$	$D^{(o,d),3}$	$D^{(o,d),4}$	$D^{(o,d),5}$	$D^{(o,d),6}$	$D^{(o,d),7}$
2000	1500	2000	3000	2000	2000	2500

According to the used tolling scheme we distinguish 6 different games:

- Game 1 (G_1): SG with uniform toll - $\theta_{l_1}^k = \theta_{l_1} \in \mathbb{R}_+^0$ for all $k \in \mathcal{K}$.
- Game 2 (G_2): SG with time-varying toll - $\theta_{l_1}^k \in \mathbb{R}_+^0$ for all $k \in \mathcal{K}$.
- Game 3 (G_3): ISG with $\theta_{l_1}^k = \xi x_{l_1}^k$, $\xi \in \mathbb{R}_+^0$.
- Game 4 (G_4): ISG with $\theta_{l_1}^k = \mu_k x_{l_1}^k$, $\mu_k \in \mathbb{R}_+^0$ for all $k \in \mathcal{K}$.
- Game 5 (G_5): ISG with $\theta_{l_1}^k = \lambda q_{l_1}^k$, $\lambda \in \mathbb{R}_+^0$.
- Game 6 (G_6): ISG with $\theta_{l_1}^k = \chi_k q_{l_1}^k$, $\chi_k \in \mathbb{R}_+^0$ for all $k \in \mathcal{K}$.

In all these games the road authority minimizes the total travel time of the network

$$\mathcal{J}_{G_i} = \sum_{k \in \mathcal{K}} \sum_{j \in \{1,2\}} \tau_{l_j}^k \, q_{l_j}^k.$$

4.1 Game 1

We are looking for constant $\theta_{l_1}^*$, where

$$\theta_{l_1}{}^* = \arg \min_{\theta_{l_1} \geq 0} \sum_{k=1}^{7} \sum_{j=1}^{2} \tau_{l_j}(\theta_{l_1}) \, q_{l_j}(\theta_{l_1}).$$

In Table 4.1 you can see the resulting functions $\tau_{l_1}^k(\theta_{l_1})$, $q_{l_1}^k(\theta_{l_1})$, $\tau_{l_2}^k(\theta_{l_1})$, $q_{l_2}^k(\theta_{l_1})$. The total travel time function $\mathcal{J}_{G_1}$ can be described as

$$\mathcal{J}_{G_1} = \frac{500}{27} \, \theta_{l_1}{}^2 - \frac{3100}{27} \, \theta_{l_1} + \frac{145000}{3},$$

and is smooth, twice continuously differentiable, and strictly convex with respect to θ_{l_1} on $\mathbb{R}_+^0$.

Table 2: Link travel times and link flows - Game 1

k	$\tau_{l_1}^k(\theta_{l_1})$	$q_{l_1}^k(\theta_{l_1})$	$\tau_{l_2}^k(\theta_{l_1})$	$q_{l_2}^k(\theta_{l_1})$
1	$2-\frac{1}{30}\,\theta_{l_1}$	$2000-\frac{200}{3}\,\theta_{l_1}$	$2+\frac{1}{15}\,\theta_{l_1}$	$\frac{200}{3}\,\theta_{l_1}$
2	$\frac{5}{2}-\frac{1}{30}\,\theta_{l_1}$	1000	$\frac{5}{2}+\frac{1}{15}\,\theta_{l_1}$	500
3	$\frac{5}{2}-\frac{1}{90}\,\theta_{l_1}$	$2000-\frac{200}{9}\,\theta_{l_1}$	$\frac{5}{2}+\frac{4}{45}\,\theta_{l_1}$	$\frac{200}{9}\,\theta_{l_1}$
4	$\frac{7}{2}-\frac{1}{30}\,\theta_{l_1}$	$2000-\frac{400}{9}\,\theta_{l_1}$	$\frac{7}{2}+\frac{1}{15}\,\theta_{l_1}$	$1000+\frac{400}{9}\,\theta_{l_1}$
5	$\frac{11}{3}-\frac{1}{30}\,\theta_{l_1}$	$\frac{4000}{3}$	$\frac{11}{3}+\frac{1}{15}\,\theta_{l_1}$	$\frac{2000}{3}$
6	$\frac{11}{3}-\frac{1}{30}\,\theta_{l_1}$	$2000-\frac{200}{9}\,\theta_{l_1}$	$\frac{11}{3}+\frac{1}{15}\,\theta_{l_1}$	$\frac{200}{9}\,\theta_{l_1}$
7	$\frac{25}{6}-\frac{1}{270}\,\theta_{l_1}$	$2000-\frac{800}{27}\,\theta_{l_1}$	$\frac{25}{6}+\frac{13}{135}\,\theta_{l_1}$	$500+\frac{800}{27}\,\theta_{l_1}$

Remark 4.1 *Function $\mathcal{J}_{G_1}$ is very simple. In some of the case studies we will not show the formulation of the total travel time function, if this formulation is too complex or long.*

The optimal value of toll is $\theta_{l_1}^* = \frac{31}{10}$ euro. With this toll value the total travel will be $\frac{1300195}{27} \approx 48155.37$ time units.

4.2 Game 2

In each time interval $k \in \mathcal{K}$ we are looking for

$$({\theta_{l_1}}^k)^* = \arg\min_{\theta_{l_1}^k \geq 0} \sum_{j=1}^{2} \tau_{l_j}(\theta_{l_j}^k)\, q_{l_j}(\theta_{l_j}^k).$$

Similarly as in Section 4.1 we can compute the resulting functions $\tau_{l_1}^k(\theta_{l_1}^k)$, $q_{l_1}^k(\theta_{l_1}^k)$, $\tau_{l_2}^k(\theta_{l_1}^k)$, $q_{l_2}^k(\theta_{l_1}^k)$. The total travel time function $\mathcal{J}_{G_2}$ is smooth, twice continuously differentiable, strictly convex with respect to each $\theta_{l_1}^k$ ($k \in \mathcal{K}$), and can be written as

$$\begin{aligned}\mathcal{J}_{G_2} = {} & \frac{444500}{9} - \frac{200}{3}\,\theta_{l_1}^1 + \frac{20}{3}\,(\theta_{l_1}^3)^2 - \frac{100}{3}\,\theta_{l_1}^2 + \frac{20}{3}\,(\theta_{l_1}^3)^2 \\ & - \frac{650}{9}\,\theta_{l_1}^3 + \frac{20}{3}\,(\theta_{l_1}^3)^2 - \frac{125}{9}\,\theta_{l_1}^4 + \frac{20}{3}\,(\theta_{l_1}^4)^2 - \frac{425}{18}\,\theta_{l_1}^5 \\ & + \frac{20}{3}\,(\theta_{l_1}^5)^2 - \frac{575}{12}\,\theta_{l_1}^6 + \frac{20}{3}\,(\theta_{l_1}^6)^2 - \frac{575}{24}\,\theta_{l_1}^7 + \frac{20}{3}\,(\theta_{l_1}^7)^2.\end{aligned}$$

The optimal toll values are $(\theta_{l_1}^1)^* = 5$, $(\theta_{l_1}^2)^* = \frac{5}{2}$, $(\theta_{l_1}^3)^* = \frac{65}{12}$, $(\theta_{l_1}^4)^* = \frac{25}{24}$, $(\theta_{l_1}^5)^* = \frac{85}{48}$, $(\theta_{l_1}^6)^* = \frac{115}{32}$, and $(\theta_{l_1}^7)^* = \frac{115}{64}$ euro. With these tolls the total travel time of the system will be $\frac{450194125}{9216} \approx 48849.19$ time units.

4.3 Game 3

We are looking for constant $\xi^* \geq 0$, where $\theta_{l_1}^k = \xi x_{l_1}^k$ for all $k \in \mathcal{K}$ and

$$\xi^* = \arg\min_{\xi \geq 0} \sum_{k=1}^{7} \sum_{j=1}^{2} \tau_{l_j}^k(\xi)\, q_{l_j}^k(\xi).$$

Table 3: Link travel times and link flows on link l_1 - Game 3

k	$\tau_{l_1}^k(\xi)$	$q_{l_1}^k(\xi)$
1	$\frac{2\,(3+100\,\xi)}{3+200\,\xi}$	$\frac{6000}{3+200\,\xi}$
2	$\frac{5(3+80\,\xi)}{2(3+200\,\xi)}$	$\frac{3000}{3+200\,\xi}$
3	$\frac{5\left(9+1000\,\xi+16000\,\xi^2\right)}{2(3+200\,\xi)^2}$	$\frac{2000\,(9+1000\,\xi)}{(3+200\,\xi)^2}$
4	$\frac{400\,\xi+21}{6+400\,\xi}$	$\frac{2000\,(200\,\xi+9)}{(3+200\,\xi)^2}$
5	$\frac{200\,\xi+11}{3+200\,\xi}$	$\frac{4000}{3+200\,\xi}$
6	$\frac{200\,\xi+11}{3+200\,\xi}$	$\frac{2000\,(9+1000\,\xi)}{(3+200\,\xi)^2}$
7	$\frac{25(9+1184\,\xi+52800\,\xi^2+64\cdot 10^4\,\xi^3)}{2(3+200\,\xi)^3}$	$\frac{1000\,(27-400\,\xi-4\cdot 10^4\,\xi^2)}{(3+200\,\xi)^3}$

In Table 4.3 you can see the resulting functions $\tau_{l_1}^k(\xi)$ and $q_{l_1}^k(\xi)$.

The total travel time function is smooth and twice continuously differentiable with respect to $\xi \in \mathbb{R}_+^0$, with one inflex point on $\mathbb{R}_+$, and can be written as

$$\begin{aligned}\mathcal{J}_{G_3} &= \frac{5000\,(7047 + 505728\cdot 10^3\,\xi^2 + 316256\cdot 10^7\,\xi^4 + 104384\cdot 10^9\,\xi^5)}{(3+200\,\xi)^6} \\ &+ \frac{5000\,(14592\cdot 10^{11}\,\xi^6 + 2765880\,\xi + 52336\cdot 10^6\,\xi^3)}{(3+200\,\xi)^6}.\end{aligned}$$

The optimal value of ξ is $\xi^* \approx 0.52\cdot 10^{-5}$. Then the total travel time of the system will be approximately 48242.37 time units and tolls imposed during individual intervals are 1.01, 1.51, 1.53, 2.52, 2.68, 2.68, and 3.15 euro, respectively.

Remark 4.2 *Note that the reason why we did not improve the outcome by use of ISG is quite straightforward. The travel time on link l_1 can be expressed as $\tau_{l_1}^k = \beta_{l_1} x_{l_1}^k + \delta_{l_1}$ and toll on link l_1 can be defined as $\theta_{l_1}^k = \xi x_{l_1}^k$, then the cost function on the link l_1 is*

$$\begin{aligned}c_{l_1}^k &= \alpha\tau_{l_1}^k + \theta_{l_1}^k = \alpha\beta_{l_1} x_{l_1}^k + \alpha\delta_{l_1} + \xi x_{l_1}^k \\ &= (\alpha\beta_{l_1} + \xi)\, x_{l_1} + \alpha\delta_{l_1}.\end{aligned}$$

The last expression is equivalent to $\tilde{c}_{l_1}^k = \alpha\tilde{\tau}_{l_1}^k$ with $\tilde{\tau}_{l_1}^k = \tilde{\beta}_{l_1} x_{l_1}^k + \delta_{l_1}$, where $\tilde{\beta}_{l_1} = \beta_{l_1} + \frac{\xi}{\alpha} \in \mathbb{R}_+$. Thus toll defined by (4.3) *imposes transformation of the bilevel optimal toll design problem into problem without tolls and link l_1 of different properties. That is why the impact of the toll is cancelled. Still, the outcome is better than the outcome of Game 2.*

4.4 Game 4

In each time interval we are looking for constant μ_k^* such that $\theta_{l_1}^k = \mu_k\, x_{l_1}^k$ and

$$\mu_k^* = \arg\min_{\mu_k \geq 0} \sum_{j=1}^{2} \tau_{l_j}^k(\mu_k)\, q_{l_j}^k(\mu_k).$$

The functions $\tau^k_{l_1}(\mu_k), q^k_{l_1}(\mu_k), \tau^k_{l_2}(\mu_k), q^k_{l_2}(\mu_k)$ can be computed as in the previous cases. The total travel time function $\mathcal{J}_{G_4}$ has a quite complex formulation, which is why we omit the formula here. It is smooth and twice continuously differentiable with respect to each μ_k. The optimal values of μ_k are $\mu^*_1 = \frac{3}{1000}$, $\mu^*_2 = \frac{3}{3400}$, $\mu^*_3 = \frac{39}{20600}$, $\mu^*_4 = \frac{3}{14200}$, $\mu^*_5 = \frac{51}{150200}$, $\mu^*_6 = \frac{23}{32600}$, $\mu^*_7 = \frac{69}{264200}$, and lead to the following tolls: 5, 1.03, 3.21, 0.44, 0.49, 1.11, and 0.47 euro, respectively. Therefore, the total travel time of the system will be $\frac{450194125}{9216} \approx 48849.19$ time units.

Remark 4.3 *Note that we can use a similar explanation as in Remark 4.2. Also, the result of this game is the same as the result of Game 2, although tolls are set differently.*

4.5 Game 5

We are looking for a constant $\lambda^* \geq 0$ such that $\theta^k_{l_1} = \lambda q^k_{l_1}$ for all $k \in \mathcal{K}$ and

$$\lambda^* = \arg\min_{\lambda \geq 0} \sum_{k=1}^{7} \sum_{j=1}^{2} \tau^k_{l_j}(\lambda)\, q^k_{l_j}(\lambda).$$

As in the previous cases we can compute functions $\tau^k_{l_1}(\lambda)$, $q^k_{l_1}(\lambda)$, $\tau^k_{l_2}(\lambda)$, and $q^k_{l_2}(\lambda)$. The total travel time function $\mathcal{J}_{G_5}$ is smooth and twice continuously differentiable on $\mathbb{R}^0_+$. The optimal value of λ is $\lambda^* \approx 0.28 \cdot 10^{-2}$. With this value of λ the total travel time of the system will be approximately 47802.67 time units and tolls imposed during individual intervals are 4.68, 3.07, 4.92, 4.96, 4.15, 4.80, and 4.65 euro, respectively.

4.6 Game 6

In each time interval we are looking for χ^*_k such that $\theta^k_{l_1} = \chi_k q^k_{l_1}$, and

$$\chi^*_k = \arg\min_{\chi_k \geq 0} \sum_{j=1}^{2} \tau^k_{l_j}(\chi_k)\, q^k_{l_j}(\chi_k).$$

The functions $\tau^k_{l_1}(\chi_k)$, $q^k_{l_1}(\chi_k)$, $\tau^k_{l_2}(\chi_k)$, $q^k_{l_2}(\chi_k)$ can be computed as in the previous cases. The total travel time function $\mathcal{J}_{G_6}$ is smooth and twice continuously differentiable with respect to each $\chi_k \in \mathbb{R}^0_+$. The optimal values of χ^k are $\chi^*_1 = \frac{3}{1000}$, $\chi^*_2 = \frac{3}{1400}$, $\chi^*_3 = \frac{39}{12200}$, $\chi^*_4 = \frac{3}{5960}$, $\chi^*_5 = \frac{51}{41800}$, $\chi^*_6 = \frac{69}{30200}$, $\chi^*_7 = \frac{2409}{736600}$ and lead to the following toll values: 5, 2.50, 5.42, 1.04, 1.77, 3.59, 6.97, respectively. The total travel time of the system will be $\frac{11650876375}{248832} \approx 46822.19$ time units. Again, the formulation of $\mathcal{J}_{G_6}$ is rather complex, which is why we do not present it here. Very interesting is that the total travel time obtained in this game is the lowest from all the performed case studies.

4.7 Discussion

In Games 5 and 6 we found a very simple tolling schemes ensuring better system performance than with the use of Stackelberg Games 1 and 2. Moreover, we showed that the outcome of Game 1 with a uniform toll is better than the outcome of Game 2 with a time-varying toll. Thus, a time-varying toll does not necessary lead to improvement of the outcome of the SG game. The outcomes of Games 5 and 6 were not better than those of Games 1 and 2. This phenomenon was explained in Sections 4.5 and 4.6. Very interesting is the fact, that Games 2 and 4 brought the same outcome. In this case the choice of tolling scheme can be driven according to some additional criteria, such as computational complexity.

5 Conclusion & Future Research

We defined the dynamic optimal toll design problem as both a Stackelberg and an inverse Stackelberg game with travelers as followers driven by a dynamic route choice equilibrium and the road authority as leader minimizing the total travel time of the system.

Moreover, on the small network inspired by the San Diego experiment (Supernak et al. (2002)) we performed six different games, of both Stackelberg and inverse Stackelberg types, and computed analytically their outcomes for the road authority. This way we showed that the road authority playing a SG can be better-off using a uniform toll than using a time-varying toll. We found out that an ISG strategy led to a better outcome for the road authority even when using very simple tolling schemes, although this does not hold for all possible tolling functions. However, we did not study more complicated tolling schemes, which might bring much better results than the ones studied in this paper. Additional research is needed also to solve large problems of the same type. For these purposes a numerical model is being developed.

The use of flow-dependent tolling is one of the possible methods how to avoid congestion on road networks. Because this approach seems to bring better results than the use of flow-independent tolls, the study of this topic can help to build more-efficient tolling systems in the future.

Acknowledgments

This research was carried out within the framework of the Next Generation Infrastructures Foundation project and the TRAIL Research School.

References

A. D. May, D. S. M. (2000) Effects of alternative road pricing systems on network performance, *Transportation research*, 34, pp. 407–436.

Başar, T., G. J. Olsder (1999) *Dynamic Noncooperative Game Theory*, SIAM, Philadelphia.

Bliemer, M. C. J. (2001) *Analytical dynamic traffic assignment with interacting usser-classes*, Ph.D. thesis, The Netherlands TRAIL Research School, Delft, The Netherlands.

Chan, K. S., W. H. K. Lam (2005) Impact of road pricing on the network reliability, *Journal of the Eastern Asia Society for Transportation Studies*, 6, pp. 2060–2075.

Joksimovič, D., M. C. J. Bliemer, P. H. L. Bovy (2004) Optimal toll design problem in dynamic trafic networks-with joint route and departure time choice, *Transportation Research Records*.

Olsder, G. J. (2005) Phenomena in inverse Stackelberg problems, in: *Regelungstheorie 11*, Mathematisches Forschungsinstitut Oberwolfach, Germany, pp. 603–605.

Patriksson, M. (1999) *Nonlinear Programming and Variational Inequality Problems: A Unified Approach*, Kluwer., The Netherlands.

Staňková, K., M. C. J. Bliemer, G. J. Olsder (2006a) Inverse Stackelberg games and their application to dynamic bilevel optimal toll design problem, in: *Proceedings of the 12th International symposium on dynamic games and applications*, INRIA, France.

Staňková, K., G. J. Olsder, M. C. J. Bliemer (2006b) Bilevel optimal toll design problem solved by the inverse Stackelberg games approach, in: *Proceedings of the 12th International conference on urban Transport and the environment in 21st century,*, WIT Press, UK, pp. 871–880.

Supernak, J., T. Golob, C. Kaschade, C. Kazimi, E. Schreffler, D. Steffey (2002) San Diego's interstate 15 congestion pricing project: Traffic-related issues, *Transportation Research Record*, 1812, pp. 43–52.

Verhoef, E. T. (2002a) Second-best congestion pricing in general networks. heuristic algorithms for finding second-best optimal toll levels and toll points, *Transportation Research Part B*, 36.

Verhoef, E. T. (2002b) Second best congestion pricing in general static transportation networks with elastic demands, *Regional Science and Urban Economics*, 32, pp. 281–310.