The period doubling bifurcation

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1 Basics

The period doubling bifurcation describes what happens when λ passes through a bifurcation value λ_0 where $F_{\lambda_0}(x) = x$ and $F'_{\lambda_0}(x) = -1$. On one side of λ_0 there is a single attracting fixed point. On the other side the attracting fixed point becomes a repelling fixed point, and an attracting periodic orbit with a 2-cycle arises.

2 Periodic doubling bifurcation theorem

In this text, we will assume that $F_\lambda(x) \stackrel{\text{def}}{=} F(x,\lambda)$ with $\lambda = 0$ has a fixed point $x_0 = 0$ with $F'_{\lambda_0}(x_0) = F'_0(0) = -1$.[1]

Note that the partial derivative of $F(x,\lambda) - x$ with respect to x is equal to -2 at $(x_0,\lambda_0) = (0,0)$. As this derivative is nonzero, so we can locally solve for x as a function of λ : there is (locally) a unique branch of fixed points, $x_0 = x_0(\lambda)$, equal to $x_0 = 0$ for $\lambda_0 = 0$. For the sake of simplicity, we will write x_0 instead of $x_0(\lambda)$, bearing in mind that x_0 is in fact λ-dependent.

Let $L(\lambda)$ be the derivative of F_λ with respect to x at the fixed point, i.e., let

$$L(\lambda) \stackrel{\text{def}}{=} \frac{\partial F}{\partial x}(x_0,\lambda).$$

Let

$$F^2(x,\lambda) \stackrel{\text{def}}{=} F^2_\lambda(x).$$

Notice that

$$\left(F^2_\lambda\right)'(x) = F'_\lambda\left(F_\lambda(x)\right)\cdot F'_\lambda(x)$$

by the chain rule. Therefore,

$$(F^2_\lambda)''(x) = F''_\lambda\left(F_\lambda(x)\right)\left(F'_\lambda(x)\right)^2 + F'_\lambda\left(F_\lambda(x)\right)F''_\lambda(x), \tag{2.1}$$

which is equal to zero for $x = x_0 = 0$ and $\lambda = \lambda_0 = 0$. In other words,

$$\frac{\partial^2 F^2}{\partial x^2}(x_0,\lambda_0) = \frac{\partial^2 F^2}{\partial x^2}(0,0) = 0. \tag{2.2}$$

[1] In fact, with a little "abuse" of notation, in this text we will freely interchange $F_\lambda(x)$ and $F(x,\lambda)$.

Now we will construct the hypothesis about second and third partial derivatives of F_λ (with respect to x) while we have to make sure that $L(\lambda)$ really passes through -1, i.e.,[2]

$$\frac{\mathrm{d}L}{\mathrm{d}\lambda}(0) \neq 0.$$

To understand what hypothesis regarding the derivatives of F_λ makes sense to make, we will differentiate (2.1) once more with respect to x:

$$\begin{aligned}\left(F_\lambda^2\right)'''(x) = {} & F_\lambda'''\left(F_\lambda(x)\right)\left(F_\lambda'(x)\right)^3 + 2F_\lambda''\left(F_\lambda(x)\right)F_\lambda''(x)F_\lambda'(x) \\ & + F_\lambda''\left(F_\lambda(x)\right)F_\lambda'(x)F_\lambda''(x) + F_\lambda'\left(F_\lambda(x)\right)F_\lambda'''(x).\end{aligned}$$

At $(x,\lambda) = (x_0,\lambda_0) = (0,0)$ this simplifies to

$$-2\frac{\partial^3 F}{\partial x^3}(0,0) - 3\left(\frac{\partial^2 F}{\partial x^2}(0,0)\right)^2. \tag{2.3}$$

Now we can state the theorem:

Theorem 2.1 *(Period doubling bifurcation) Let $F_\lambda \in \mathscr{C}^3$ and let*

$$F_0'(0) = -1, \tag{2.4}$$

$$\frac{\mathrm{d}L}{\mathrm{d}\lambda}(0) > 0, \tag{2.5}$$

$$2\frac{\partial^3 F}{\partial x^3}(0,0) + 3\left(\frac{\partial^2 F}{\partial x^2}(0,0)\right)^2 > 0. \tag{2.6}$$

Then there are non-empty intervals $(\lambda_1,0)$ and $(0,\lambda_2)$ and $\varepsilon > 0$ so that

1. *If $\lambda \in (\lambda_1,0)$ then F_λ has one repelling fixed point and one attracting periodic orbit with a 2-cycle in $(-\varepsilon,\varepsilon)$.*
2. *If $\lambda \in (0,\lambda_2)$ then F_λ^2 has a single fixed point in $(-\varepsilon,\varepsilon)$ which is in fact an attracting fixed point of F_λ.*

Proof
The proof can be divided into 6 steps:

- **Step 1**
  Let $H(x,\lambda) \stackrel{\text{def}}{=} F_\lambda^2(x,\lambda) - x$. Then H and its first two partial derivatives with respect to x become zero at x_0. Equations (2.3) and (2.6) give

  $$\frac{\partial^3 H}{\partial x^3}(0,0) < 0.$$

  One of the roots of $H(x,\lambda) \stackrel{\text{def}}{=} F^2(x,\lambda) - x$ at $x = x_0 = 0$, $\lambda = \lambda_0 = 0$ corresponds to the fact that $(x_0,\lambda_0) = (0,0)$ is a fixed point. That means that there exists $P(x,\lambda)$ such that

  $$H(x,\lambda) = (x - x_0)P(x,\lambda). \tag{2.7}$$

[2] Note that for the class of functions that we are dealing with during this course this condition is always satisfied.

- **Step 2**
  Then

$$\frac{\partial H}{\partial x}(x,\lambda) = P + (x-x_0)\frac{\partial P}{\partial x}(x,\lambda),$$
$$\frac{\partial^2 H}{\partial x^2}(x,\lambda) = 2\frac{\partial P}{\partial x}(x,\lambda) + (x-x_0)\frac{\partial^2 P}{\partial x^2}(x,\lambda),$$
$$\frac{\partial^3 H}{\partial x^3}(x,\lambda) = 3\frac{\partial^2 P}{\partial x^2}(x,\lambda) + (x-x_0)\frac{\partial^3 P}{\partial x^3}(x,\lambda).$$

  This means that $P(0,0)=0$ and $\frac{\partial P}{\partial x}(0,0)=0$, while

$$\frac{\partial^3 H}{\partial x^3}(0,0) = 3\frac{\partial^2 P}{\partial x^2}(0,0).$$

  Thus

$$\frac{\partial^2 P}{\partial x^2}(0,0) < 0. \tag{2.8}$$

- **Step 3**
  Let us first recall the Implicit function theorem (I will state it in general form with all parts unlike at the lecture, just to refresh your memory about this theorem ☺)

  Theorem 2.2 *(Implicit function theorem) Suppose $f: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$, has $f(\alpha) = \mu$ for some $\alpha = (\alpha_1, \dots, \alpha_n) \in D$ and $\mu \in \mathbb{R}$, while $\frac{\partial f}{\partial x_n}(\alpha) \neq 0$. Then the following statements hold:*

  1. *There is a function $g(x_1, \dots, x_{n-1})$ defined near $(\alpha_1, \dots, \alpha_{n-1}) \in D \cap (\mathbb{R}^{n-1} \times \{\alpha_n\})$, such that*

$$f(x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1})) = \mu.$$

  2. *Near α, the level set*

$$L_f(\mu) \stackrel{\text{def}}{=} \{\xi \in D | f(\xi) = \mu\}$$

     is an $(n-1)$-dimensional manifold, and its tangent plane at α is perpendicular to $\nabla f(\alpha)$.

  3. *The derivative of g at $(\alpha_1, \dots, \alpha_{n-1})$ is given by*

$$g'(\alpha_1, \dots, \alpha_{n-1}) = \left[\frac{\partial g}{\partial x_1} \cdots \frac{\partial g}{\partial x_{n-1}}\right](\alpha_1, \dots, \alpha_{n-1})$$
$$= \left[-\frac{\frac{\partial f}{\partial x_1}(\alpha)}{\frac{\partial f}{\partial x_n}(\alpha)} \cdots -\frac{\frac{\partial f}{\partial x_{n-1}}(\alpha)}{\frac{\partial f}{\partial x_n}(\alpha)}\right].$$

Now, we would like to apply the implicit value theorem to $P(x,\lambda) = 0$ and solve this equation for λ as a function of x. This will allow us to determine the fixed points of F_λ^2 that are not fixed points of F_λ, i.e., the periodic points with a 2-cycle. For being able to apply Theorem 2.2, we have to show that[3]

$$\frac{\partial P}{\partial \lambda}(0,0) < 0. \tag{2.9}$$

To prove (2.9) we compute $\frac{\partial H}{\partial x}$ both from its definition $H(x,\lambda) \stackrel{\text{def}}{=} F^2(x,\lambda) - x$ and from (2.7) and we obtain

$$\begin{aligned}\frac{\partial H}{\partial x}(x,\lambda) &= \frac{\partial F}{\partial x}(F(x,\lambda),\lambda)\ \frac{\partial F}{\partial x}(x,\lambda) - 1 \\ &= P(x,\lambda) + (x - x_0)\frac{\partial P}{\partial x}(x,\lambda).\end{aligned}$$

Recall that x_0 is the fixed point of F_λ and that

$$L(\lambda) \stackrel{\text{def}}{=} \frac{\partial F}{\partial x}(x_0,\lambda).$$

So substituting $x = x_0$ into the preceding equation gives

$$(L(\lambda))^2 - 1 = P(x,\lambda).$$

Differentiating this expression with respect to λ and setting $\lambda = 0$ gives

$$\frac{\partial P}{\partial \lambda}(0,0) = 2L(0)L'(0) = -2L'(0),$$

which is negative by assumption (2.5).

- **Step 4**
  By the implicit function theorem, (2.9) implies that there is a $\mathscr{C}^2$-function $g(x)$ defined near $x = x_0 = 0$, being the unique solution of $P(x,g(x)) \equiv 0$. Recall that $P(x,\lambda)$ and its first derivative with respect to x become zero at $(x,\lambda) = (x_0,\lambda_0) = (0,0)$. Also, according to implicit function theorem,

  $$g'(x) = -\frac{\frac{\partial P}{\partial x}}{\frac{\partial P}{\partial \lambda}}.$$

  Therefore,

  $$g'(0) = 0$$

  and

  $$g''(0) = -\frac{\frac{\partial^2 P}{\partial x^2}}{\frac{\partial P}{\partial \lambda}}(0,0) < 0,$$

[3] Or that

$$\frac{\partial P}{\partial \lambda}(0,0) > 0.$$

since both nominator and denominator are negative. As $g'(0) = 0$ and $g''(0) < 0$, $g(x)$ has a maximum at $x = x_0 = 0$, and this maximum value is 0. In the (x, λ)-plane. the graph of $g(x)$ looks locally approximately like a parabola in the lower half plane with its vertex at the origin. Clearly, there are no fixed points for positive $g(x)$ and two fixed points for $g(x) < 0$.

- **Step 5**
  Condition (2.5)
  $$\frac{\mathrm{d}L}{\mathrm{d}\lambda}(0) > 0$$
  and the fact that $L(0) = -1$ imply that $L(\lambda) < -1$ for $\lambda < 0$ and $L(\lambda) > -1$ for $\lambda > 0$, so the fixed point x_0 is repelling to the left and attracting to the right. As for the 2-periodic points, we wish to show that
  $$\frac{\partial F^2}{\partial x}(x, g(x)) < 1$$
  for $x < 0$. Now (2.2) and $g'(0) = 0$ imply that 0 is a critical point for this function and the value at this critical point is $(L(0))^2 = 1$. To complete the proof we must show that the critical point is a local maximum. So we must compute the second derivative at $x_0 = 0$..

- **Step 6**
  Defining
  $$\begin{aligned}
  \phi(x) &\stackrel{\text{def}}{=} \frac{\partial F^2}{\partial x}(x, g(x)),\\
  \phi'(x) &= \frac{\partial^2 F^2}{\partial x^2}(x, g(x)) + \frac{\partial^2 F^2}{\partial x \partial \lambda}(x, g(x))\, g'(x),\\
  \phi''(x) &= \frac{\partial^3 F^2}{\partial x^3}(x, g(x)) + 2\,\frac{\partial^3 F^2}{\partial x^2 \partial \lambda}(x, g(x))\, g'(x)\\
  &\quad + \frac{\partial^3 F^2}{\partial x \partial \lambda^2}(x, g(x))\, (g'(x))^2 + \frac{\partial^2 F^2}{\partial x \partial \lambda}(x, g(x))\, g''(x).
  \end{aligned}$$
  Since $g'(0) = 0$, the middle two terms in the last expression become 0 and the last term becomes
  $$\frac{\mathrm{d}L}{\mathrm{d}\lambda}(0)\, g''(0) < 0$$
  by condition (2.5) and the fact that $g''(0) < 0$. We have computed the first term, i.e. the third partial derivative, in (2.3) using condition (2.4) and then (2.6) implies that this expression is negative. This completes the proof of the period doubling bifurcation theorem.

□

3 Concluding notes

There are variants of Theorem 2.1 with signs in (2.5) and (2.6). Therefore, we may have an attracting fixed point merging into two repelling points of period two to produce a repelling fixed point, and/or the direction of the bifurcation may be reversed. As discussed during the lecture, condition (2.6) can be replaced by a condition involving higher derivatives of $F_\lambda(x)$ with respect to x.