Lecture 12: Some Important Continuous Probability Distributions (Part 1)

Kateřina Staňková

Statistics (MAT1003)

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Outline

1. **Uniform Distribution**
   - Formulation
   - Expectation & Variance
   - Examples

2. **Flavor of estimation problems . . .**

3. **Exponential Distribution**
   - Formulation
   - Expectation etc.

book: Sections

And now . . .

Formulation

What is uniform PDF?

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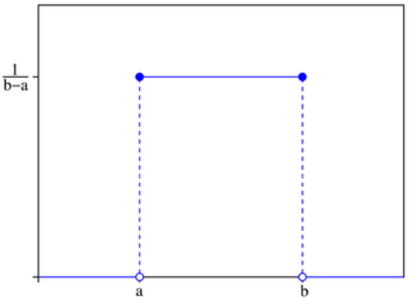

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$$= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_{x=a}^{b}$$

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$$V(X) = E(X^2) - \mu_x^2 = \ldots = \frac{1}{12}(b-a)^2 \quad \text{(check yourself)}$$

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- So for the density we have $f(z) = \begin{cases} 0, & z < 0 \\ 2z, & 0 \le z \le 1 \\ 0, & z > 1 \end{cases}$ (not uniform)
- Hence $E(Z) = \int_0^1 z \cdot 2z \, \mathrm{d}z = \left[\frac{2}{3} z^3\right]_{z=0}^1 = \frac{2}{3}$

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    $= \int_0^z \frac{z-y}{10} \cdot \frac{1}{10} \, \mathrm{d}y$

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  - Case $z \in [10, 20]$
    $= \int_0^{z-10} \frac{1}{10} \, \mathrm{d}y + \int_{z-10}^{10} \frac{z-y}{10} \cdot \frac{1}{10} \, \mathrm{d}y$

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    $= \int_0^{z-10} \frac{1}{10}\,\mathrm{d}y + \int_{z-10}^{10} \frac{z-y}{10} \cdot \frac{1}{10}\,\mathrm{d}y =$
    $\left[\frac{1}{10}(z-10) + \frac{1}{100}(z\,y - \frac{y^2}{2})\right]_{y=z-10}^{10} = \ldots =$
    $\frac{1}{100}(20\,z - \frac{z^2}{2} - 100)$

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$$f(z) = \begin{cases} 0, & z < 0, \\ \frac{z}{100}, & 0 \leq z \leq 10 \\ \frac{1}{100}(20 - z), & 10 \leq z \leq 20 \\ 0, & z > 20 \end{cases}$$

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- We call $X_1, X_2, \ldots, X_{100}$ a random sample
- After 100 drawings we have 100 realizations, denoted by $X_1, X_2, \ldots, X_{100}$

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- Then $E(Z) = \int_0^L z \cdot 100 \cdot \left(\frac{z}{L}\right)^{99} \cdot \frac{1}{L}\, \mathrm{d}z = 100 \int_0^L \left(\frac{z}{L}\right)^{100} \mathrm{d}z$
  $= \left[100 \cdot \frac{1}{101} \left(\frac{z}{L}\right)^{101} \cdot L\right]_{z=0}^{L} = \frac{100}{101} L$

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  $= \left[100 \cdot \frac{1}{101}\left(\frac{z}{L}\right)^{101} \cdot L\right]_{z=0}^{L} = \frac{100}{101}L$
- As an estimate for L we now define the RV $B = \frac{101}{100} \cdot Z$. We have $E(B) = L$
- B is called an unbiased estimator for L

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Observation 1

- $E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n} \mu_X = \frac{1}{n} \cdot n \cdot \mu_X = \mu_X$

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Observation 2

Let $X_1, \ldots, X_n$ be IID with $\mu = E(X)$, $\sigma^2 = V(X)$. Then

$$V(\bar{X}) = V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = V\left(\frac{1}{n}X_1 + \frac{1}{n}X_2 + \ldots + \frac{1}{n}X_n\right)$$

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$$= \frac{1}{n^2}V(X_1) + \frac{1}{n^2}V(X_2) + \ldots + \frac{1}{n^2}V(X_n)$$

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Notice that

$$\underbrace{V(\bar{X}) = E\left\{(X-\mu)^2\right\}}_{\text{variance of sample mean}} \neq \underbrace{S^2 \approx E\left\{(X_i - \bar{X})^2\right\}}_{\text{sample variance}}$$

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Moreover, Observation 2 is independent of the actual distribution of the X_i

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Let X have the following distribution: $P(X = 1) = \bar{p}$, $P(X = 0) = 1 - \bar{p}$ with $\bar{p}$ unknown (0 elsewhere). Estimate $\bar{p}$

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    HW: What is the probability distribution of $\sum_{i=1}^{n} X_i$?

And now . . .

Formulation

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A RV X is exponentially distributed with parameter λ (book: parameter $1/\beta$), $X \sim \mathrm{Exp}(\lambda)$, if

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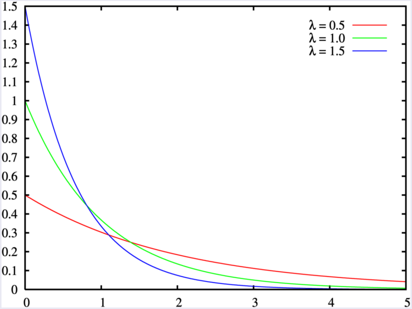

Computation of $E(X)$, $F(X)$ and $V(X)$ for exponential distribution

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