Lecture 4: Probabilities and distributions (Part 2)

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Statistics (MAT1003)

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Outline

And now . . .

1. A lot of exercises from last topic. . .
2. Discrete Joint PD
3. Discrete Marginal Probability Distribution
4. Continuous Joint PD

Please check your notes from this lecture, focus in exercises was on all new notions from previous lecture ...

Not so many new things were dealt with in the rest of the lecture ...

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And now . . .

book: Section 3.4

Discrete Joint Probability Distribution (Def. 3.8)

For two discrete RV's X and Y the joint probability distribution f is defined as follows:

$$f(x, y) = P(X = x, Y = y)$$

Example 3.14

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picking $2-x-y$ green: $\binom{3}{2-x-y}$, picking any 2: $\binom{8}{2}$

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$$f(x, y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}},$$

$x \in \{0, 1, 2\}$, $y \in \{0, 1, 2\}$, $0 \leq x + y \leq 2$.

And now . . .

Discrete Marginal Probability Distribution

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$$g(x) = P(X = x) = \sum_{y} P(X = x, Y = y) = \sum_{y} f(x, y)$$

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For the PDF h of Y we have

$$h(y) = P(Y = y) = \sum_{x} P(X = x, Y = y) = \sum_{x} f(x, y)$$

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Possible pairs (x, y) are $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$, $(0, 2)$, and $(2, 0)$.

$$P(X = 1) = P(X = 1, Y = 0) + P(X = 1, Y = 1)$$

$$= \frac{\binom{3}{1}\binom{2}{0}\binom{3}{1}}{\binom{8}{2}} + \frac{\binom{3}{1}\binom{2}{1}\binom{3}{0}}{\binom{8}{2}} = \frac{15}{28}$$

And now . . .

book: Section 3.4

Continuous Joint Probability Distribution

Let X be a continuous RV with probability density function f. Then

$$f(x) = F'(x) = \frac{\mathrm{d}}{\mathrm{d}x} P(X \leq x)$$
$$= \lim_{\Delta x \to 0} \frac{P(X \leq x + \Delta x) - P(X \leq x)}{\Delta x}$$
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Analogously for 2 continuous RV's X and Y :

$$f(x,y) = \lim_{(\delta,\epsilon)\to(0,0)} \frac{P(x \leq X \leq x+\delta, y \leq Y \leq y+\epsilon)}{\delta\epsilon}$$

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Properties of Continuous Joint PD

1. $f(x,y) \ge 0$ for all $x, y \in \mathbb{R}$
2. $\iint_{\mathbb{R}^2} f(x,y)\,dxdy = 1$
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Example 1

X, Y RV's with

$$f(x,y) = \begin{cases} \frac{y}{2x^2}, & \frac{1}{2} < x < 1,\ 0 < y < 2 \\ 0, & elsewhere \end{cases}$$

(a) Validate condition 2. (b) Calculate $P(X \geq \frac{3}{5})$

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$$= \int_{\frac{1}{2}}^{1}\left[\frac{y^2}{4x^2}\right]_0^2 \mathrm{d}x = \int_{\frac{1}{2}}^{1}\frac{1}{x^2}\mathrm{d}\,x = -\left[\frac{1}{x}\right]_{\frac{1}{2}}^{1} = 1$$

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Example 1(b): Calculate $P(X \geq \frac{3}{5})$

$$P(X \geq \frac{3}{5}) = \int_{\frac{3}{5}}^{1}\int_{0}^{2} \frac{y}{2\,x^2}\mathrm{d}y\mathrm{d}x = \int_{\frac{3}{5}}^{1} \frac{1}{x^2}\mathrm{d}\,x = \frac{2}{3}$$