Lecture 7: Bayes′ Rule, Revision

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Statistics (MAT1003)

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Outline

1. **Preliminaries**
   - Partition of a sample space
2. **Bayes′ rule**
   - Theory
   - Examples
3. **A lot of computing . . .**

And now . . .

Partition of a sample space

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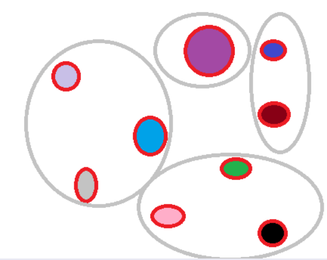

Partition of a sample space

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Any sample space S

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Example 3

Any sample space S
Partition $\{B, B'\}$ for any $B \subseteq S$

Observation

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Let $B_1, \ldots, B_k$ be a partition of S. Then for any $A \subseteq S$:

$$\begin{aligned} A = A \cap S &= A \cap (B_1 \cup B_2 \cup \ldots \cup B_k) \\ &= \underbrace{(A \cap B_1)}_{A_1} \cup \underbrace{(A \cap B_2)}_{A_2} \cup \ldots \cup \underbrace{(A \cap B_k)}_{A_k} \end{aligned}$$

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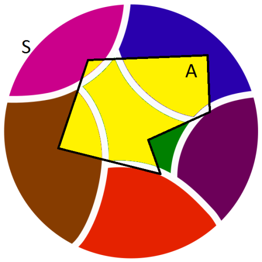

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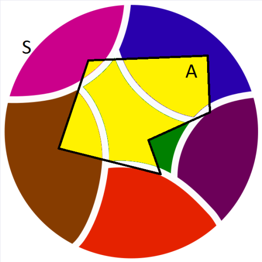
$P(A_i) = P(A \cap B_i) = P(B_i) \cdot P(A|B_i)$

$A_1, \ldots, A_k$ disjoint sets



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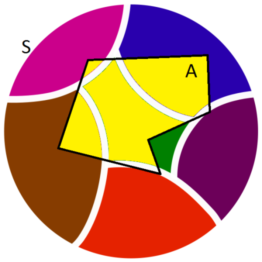

$P(A_i) = P(A \cap B_i) = P(B_i) \cdot P(A|B_i)$

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$$P(A) = \sum_{i=1}^{k} P(A \cap B_i)$$
$$= \sum_{i=1}^{k} P(B_i) \cdot P(A|B_i)$$

And now ...

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Exercise 2.95 + forget about 40

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- Data:
  - $P(C) = 0.05 \quad (P(C') = 0.95)$

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  Complementary events: C' and D'
- Data:
  - $P(C) = 0.05 \quad (P(C') = 0.95)$
  - $P(D|C) = 0.78 \quad (P(D'|C) = 0.22)$

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- Question 1: $P(D) = ?$

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- Question 1:$P(D) =?$
- Solution:

$$P(D) =$$

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- Question 1:$P(D) =?$
- Solution:

$$P(D) = P(D \cap C) + P(D \cap C')$$

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- Question 1: $P(D) = ?$
- Solution:

$$\begin{aligned} P(D) &= P(D \cap C) + P(D \cap C') \\ &= P(C) \cdot P(D|C) + P(C') \cdot P(D|C') \end{aligned}$$

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- Question 1: $P(D) =?$
- Solution:

$$\begin{aligned} P(D) &= P(D \cap C) + P(D \cap C') \\ &= P(C) \cdot P(D|C) + P(C') \cdot P(D|C') \\ &= 0.05 \cdot 0.78 + 0.95 \cdot 0.06 = 0.039 + 0.054 = 0.093 \end{aligned}$$

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- Solution:

$$P(C|D) =$$

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- Solution:

$$P(C|D) = \frac{P(C \cap D)}{P(D)}$$

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- Solution:

$$P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{P(D|C) \cdot P(C)}{P(D)} = \frac{0.78 \cdot 0.05}{0.093} = \frac{0.039}{0.093} = \frac{13}{31}$$

Exercise 2.95 - Question 3

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- Question 3: What is the probability that someone who is diagnosed as not having cancer actually has the disease?

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$$P(C|D') =$$

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- Question 3: What is the probability that someone who is diagnosed as not having cancer actually has the disease? $P(C|D') = ?$
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$$P(C|D') = \frac{P(C \cap D')}{P(D')}$$

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- Question 3: What is the probability that someone who is diagnosed as not having cancer actually has the disease? $P(C|D') = ?$
- Solution:

$$P(C|D') = \frac{P(C \cap D')}{P(D')} = \frac{P(D'|C) \cdot P(C)}{P(D')}$$
$$= \frac{0.05 \cdot 0.22}{0.907} = \frac{0.0011}{0.907} \approx 0.012$$

And now . . .

1. Preliminaries
   - Partition of a sample space
2. Bayes′ rule
   - Theory
   - Examples
3. A lot of computing . . .

- Try all odd exercises from Section 2.7 (pp. 76–77), Exercise 2.99 was done in the class
- Also, you should be able to compute all Review exercises from pp. 77-79, check it!