Lecture 9: Variance, Covariance, Correlation Coefficient

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Statistics (MAT1003)

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Outline

1. **Variance**
   - Definition
   - Standard Deviation
   - Variance of linear combination of RV
2. **Covariance**
   - Meaning & Definition
   - Examples
3. **Correlation coefficient**

book: Sections 4.2, 4.3

And now . . .

1. Variance
   - Definition
   - Standard Deviation
   - Variance of linear combination of RV
2. Covariance
   - Meaning & Definition
   - Examples
3. Correlation coefficient

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Alternative formula

$$\begin{aligned} V(X) &= E\{(X - \mu_X)^2\} = E(X^2 - 2\,\mu_X\,X + \mu_X^2) \\ &= E(X^2) - 2\,\mu_X\,E(X) + \mu_X^2 = E(X^2) - 2\,\mu_X^2 + \mu_X^2 \\ &= E(X^2) - \mu_X^2 \end{aligned}$$

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Observation

Variance is always nonnegative!

Standard deviation σ_X

Square root of $V(X)$, i.e., $\sigma_X = +\sqrt{V(X)} = +\sqrt{\underbrace{E(X^2) - \mu_X^2}_{V(X)}}$

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$$V(X) = E(X^2) - \mu_X^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

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n : a real number

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$$V(n) = E(n^2) - \mu_n^2 = n^2 - n^2 = 0, \quad \sigma_n = 0$$

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$$g(x) = \frac{10}{3}x - \frac{10}{3}x^4, \quad 0 < x < 1 \quad \text{(0 elsewhere)}$$

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Theorem about $V(aX + b)$

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Let X be an RV with variance $V(X)$. Then:
$V(a\,X + b) = a^2\,V(X)$

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Proof:

$$\begin{aligned} V(aX + b) &= E\{(aX + b - \mu_{aX+b})^2\} \\ &= E\{(aX + b - (a\mu_X + b))^2\} \\ &= E\{(aX - a\mu_X)^2\} = E\{a^2(X - \mu_X)^2\} \\ &= a^2 E\{(X - \mu_X)^2\} = a^2 V(X) \end{aligned}$$

□

And now . . .

1 Variance
- Definition
- Standard Deviation
- Variance of linear combination of RV

2 Covariance
- Meaning & Definition
- Examples

3 Correlation coefficient

What is covariance?

Dependence of realizations of 2 (or more) different RVs.

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Definition

Let X and Y be RVs with $\mu_X = E(X)$ and $\mu_Y = E(Y)$. Then

$$\mathrm{Cov}(X, Y) = E\{(X - \mu_X)(Y - \mu_Y)\}$$

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Remark

Covariance can be positive and negative (variance is always nonnegative)

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Alternative formula

$$\begin{aligned}
\mathrm{Cov}(X, Y) &= E\{(X - \mu_X)(Y - \mu_Y)\} \\
&= E\left(XY - \mu_Y \cdot X - \mu_X \cdot Y + \mu_X \mu_Y\right) \\
&= E(XY) - \mu_Y \cdot \mu_X - \mu_X \cdot \mu_Y + \mu_X \mu_Y \\
&= E(XY) - \mu_X \mu_Y
\end{aligned}$$

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Example 1

X, Y independent RVs

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$$\mathrm{Cov}(X, Y) = E(XY) - \mu_X \mu_Y = E(X) \cdot E(Y) - \mu_X \cdot \mu_Y = 0$$

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Example 2

X, Y independent RVs with joint PDF

$$f(x, y) = \begin{cases} 10x\,y^2, & 0 < x < y < 1 \\ 0, & \text{elswhere} \end{cases} .$$

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- $\rho(X, Y) = \frac{\frac{5}{378}}{\sqrt{\frac{55}{1134}} \cdot \sqrt{\frac{5}{252}}} \approx 0.4264$